A statistical learning approach for constrained sensor placement

Kévin Kasper¹, Lionel Mathelin² and Hisham Abou-Kandil³

Abstract—Sensor placement is of pivotal importance in closed-loop control as measurements are key to design the control laws. In this article, a novel statistical learning-based sensor placement algorithm is proposed in order to recover a high-dimensional field from a limited amount of local measurements with a linear estimator. Unlike many other methods, our algorithm does not rely on a reduced order model and achieves good results even with a small number of sensors. In many situations, sensors cannot be placed arbitrarily, either because of their geometry or because of the environment they are in. Our algorithm naturally accounts for these constraints as well as being robust to noise. Its performance is illustrated on a fluid flow example and compared to two state of the art methods, Effective Independence and FrameSense, on the recovery of the pressure field from limited noisy pressure measurements.

I. INTRODUCTION

Sensor placement problems are present in numerous domains such as control, estimation, fault detection and classification [18]. Such problems are still subject to active research because of their complexity, stemming from their inherent combinatorial nature. Several factors must be considered when installing sensors such as their number, their location and the quantity of interest one wants to measure.

A wide range of efficient sensor placement methods have emerged, in particular from the active domain of structure monitoring. Some of the most popular ones are Effective Independence, Optimal Driving Point, Sensor Set Expansion, Kinetic Energy Method and Variance Method [2], [14]. Most rely on a criterion derived from the Fisher Information Matrix (FIM) and on a reduced order basis that efficiently describes the high-dimensional field. Other criteria can also be used, such as the Modal Assurance Criteria [13], the condition number of the modal shape matrix [4], or the maximization of the smallest singular value of the associated FIM [1]. If more information is known on the problem at hand, specific methods can be used. A prior knowledge can help recover the properties of a fluid flow [5], a specific model can help isotropic source localization [7] and using spline functions can help recover seismic responses [11]. All the above methods are based on a signal processing and information theory approach. Other techniques stemming from Control Theory use observability gramians [9] or Hankel Singular Values [6]. Overall, many of them can be used in many contexts after minor adjustments. The work of Kammer [8] on Effective Independence (EI) should be highlighted because of its popularity and performance. It consists in ensuring that each measurement is as independent as possible from the others in order for them to effectively contribute to the estimation goal. Efficient numerical implementation [10] can further decrease the computation time. Another remarkable method is the recent FrameSense (FS) [16]. It exploits numerous results from frame theory in order to guarantee good recovery properties. All these different criteria are then coupled with an optimization scheme. Most involve a greedy algorithm, greatly simplifying the problem by using a series of optimal local steps instead of global ones. Other popular choices include heuristic algorithms such as Genetic Algorithms [3] and Simulated Annealing [12].

In contrast with the above methods, the approach presented in the present paper does not rely on a given reduced-order basis to solve the sensor placement problem but on an implicit representation basis and a specific optimization problem exploiting a set of measured data. This formulation is flexible, accounts for prior noise information and can be tailored to specific recovery objectives if the sensors have to satisfy constraints for their placement.

In this article, we focus on the recovery of a high-dimensional field from a limited number of spatially constrained sensors in order to improve control performances. The sensors output the local value of the field to be recovered, plagued with some noise. The recovery problem is solved using statistical learning: from a set of prior data available on the field and information on the sensors placement constraints and inherent noise, the sensors locations will be learnt together with a suitable linear estimator. To illustrate our approach, the relative pressure field of a fluid flow around a cylinder is estimated from a limited number of surface-mounted pressure sensors.

The paper is organized as follows. In section 2, the problem is formalized in a suitable framework and related work is briefly reviewed. Our algorithm is detailed in section 3 and its specificities are highlighted. Numerical results illustrating the method and a comparison with two state of the art approaches on the recovery of the pressure field around a cylinder are shown in section 4. Concluding remarks are made in section 5.

II. PROBLEM STATEMENT

A. Framework

Let \( f(x, t) \in \mathbb{R} \) be our field of interest. \( x \) is the spatial coordinate vector and \( t \) is time. To enter the numerical processing framework, a spatial and temporal discretization...
is required. The vector $x = (x_1, x_2, \ldots, x_{n_x}) \in \mathbb{R}^{n_x}$ will be finite, with $n_x$ the number of grid points. We consider discrete values of time labelled $\{t_1, \ldots, t_{n_t}\}$, with $n_t$ the number of time instants considered.

The discretized field at $t_i$ is represented by $y_i \in \mathbb{R}^{n_x}$, a vector containing the values of $f$ at each grid point and is referred to as a snapshot. From a collection of snapshots at different times, we build a learning sequence, or snapshot matrix:

$$Y = [y_1, y_2, \ldots, y_{n_t}] \in \mathbb{R}^{n_x \times n_t}. \quad (1)$$

This sequence must contain information relevant for the goal at hand. In a control context, it is important to gain information on the dynamics of the physical system through measurements both on transient and asymptotic regimes. More generally, the learning sequence must contain information as representative as possible of the actual situation.

The sensor placement problem can now be formalized as

$$\text{Find } \{C, \psi\} \in \arg \min_{C \in \mathcal{M}_C, \psi} \mathbb{E} \left[ \|Y - \boldsymbol{\psi}(CY + \Xi)\|_F \right], \quad (8)$$

with $\mathbb{E} [\cdot]$ the expectation operator. In the noiseless case, the criterion (7) simply becomes $\varepsilon = \|Y - \psi CY\|_F$ and the expectation is no longer required in (8). After illustrating our algorithm in this idealistic framework, noisy data will be seamlessly integrated.

C. Prior Work

To solve (8), nearly every technique from the literature starts by reducing the dimension of the problem. The data $Y \in \mathbb{R}^{n_x \times n_t}$ is expressed in a compressed way through a reduced order basis $\phi \in \mathbb{R}^{n_D \times n_x}$, where $n_D < n_t$ is the number of modes retained. Such bases are abusively referred to as such even though they are not spanning sets of $\mathbb{R}^{n_x}$. One then has $y_i \approx \tilde{y}_i = \phi a_i$. Let $A = [a_1, a_2, \ldots, a_{n_D}] \in \mathbb{R}^{n_D \times n_x}$ be the matrix containing the approximation of each snapshot in the basis $\phi$. If the basis is satisfactory, the description error

$$\varepsilon_d = \|Y - \tilde{Y}\|_F = \|Y - \phi A\|_F \quad (9)$$

is small in the relative sense, less than $10^{-2} \|Y\|_F$ for instance. One of the most popular dimension reduction methods is Proper Orthogonal Decomposition (POD). The Snapshot Method [17] is typically used to compute the POD basis of $Y$ of order $n_D$. Via a simple Singular Value Decomposition (SVD) of $Y$, this basis $\phi$ is obtained as the first $n_D$ left singular vectors of the SVD. It has the desirable property of being orthonormal as well as being the best $n_D$-term approximation of $Y$ in the Frobenius norm-sense. For sake of brevity, subscript $i$ will be omitted in the sequel.

With this basis, an estimator $\psi$ can be specified. The most common approach estimates $\phi$ instead of $y$ (the lower dimension simplifies the problem) by solving the following least square problem

$$\tilde{a} \in \arg \min_{\phi \in \mathbb{R}^{n_D}} \|\tilde{s} - C\phi \tilde{a}\|_2. \quad (10)$$

The full field estimate is now directly given by

$$\tilde{y} = \phi \tilde{a} = \phi (C\phi) \tilde{s}. \quad (11)$$
with \((\cdot)^+\) the Moore-Penrose pseudo-inverse operator, and the associated estimator is then
\[
\psi_{\text{POD}} = \phi (C\phi)^+.
\] (12)

Before focusing on finding \(C\) now that \(\psi\) is specified, let us specify how the number of modes \(n_D\) is chosen. A large \(n_D\) improves the description performance of the POD basis but if \(n_D > n_S\), problem (10) becomes ill-posed and \(a\) becomes unobservable. Under these conditions, one often chooses \(n_D = n_S\) so as to exploit as much sensor data as possible.

A suitable \(C\) can be found by minimizing a given criterion most often related to the FIM, involving \(\hat{\phi}^T \phi\), with \(\phi = C\phi\). The sensors are considered plagued by independent and stationary identically distributed additive Gaussian white noise. The covariance matrix of \(\hat{a}\) can be expressed as a weighted FIM that is then used in an appropriate criterion. Effective Independence [8], for instance, starts with \(\text{pos}S \leftarrow \text{pos}P\) and removes a sensor from \(\text{pos}S\) at each iteration. The noiseless case is first considered to better illustrate our algorithm. As seen previously, problem (8) is solved by minimizing \(\epsilon\) (7). For a given \(C\), the optimal estimator \(\psi_o\) minimizing \(\epsilon = \|Y - \psi CY\|_F\) is
\[
\psi_o = Y(CY)^+.
\] (13)

This is the estimator used to recover the field of interest. Note that \(\psi_o = \psi_{\text{POD}}\) in cases where \(\epsilon_d = 0\), Eq. (9), and \(n_S \geq n_D\). This does not happen in our framework where \(n_S < n_D\). This condition is more prominent when the sensor locations are constrained as more measurements are needed to overcome the suboptimal positioning and maintain recovery accuracy. With this closed-form expression for \(\psi\), the sensor matrix \(C \in \mathbb{R}^{n_S \times n_x}\) now satisfies
\[
C \in \arg \min_{C \in \mathcal{M}_C} \left\| Y - Y \left( \bar{C}Y \right)^+ \bar{C}Y \right\|_F.
\] (14)

The combinatorial nature of the solution method is overcome with a greedy approach at the price of possibly ending in a local minimum.

The optimization scheme is now detailed. The set \(\text{pos}P\) is initialized with possible sensor locations and \(\text{pos}C\) receives \(n_S\) random sensors from \(\text{pos}P\). Then, \(\epsilon\) is evaluated from Eq. (7) and the sensors are iteratively updated one at a time. A random sensor from \(\text{pos}C\) is removed and \(\text{pos}P\) is updated in order to reflect the new possible sensor locations. Spatial extension constraints are easily taken into account by banning the vicinity of the remaining sensors. Every location of \(\text{pos}P\) is then tested by adding it to \(\text{pos}C\) and evaluating the corresponding error \(\epsilon\). The location of the sensor most minimizing \(\epsilon\) is retained. This loop is repeated until \(\epsilon\) no longer decreases (local minimum found) or the reconstruction error is deemed satisfactory. This algorithm can be further improved by slightly moving some sensors in a random way to escape local minima.

To summarize, our algorithm exhibits the following specificities:

- It allows good results with \(n_S < n_D\) sensors (where \(n_D\) would be the number of POD modes ensuring a low \(\epsilon_d\)). This is made possible by keeping our focus on \(y\) instead of shifting it to the smaller \(a\) as well as using \(Y\) in the criterion and estimator instead of an \(n_D\) order POD basis.
- It accounts for placement constraints. During the iterative step, the algorithm constantly updates the possible locations and naturally accounts for sensor geometry constraints. Further, since the algorithm does not rely on reduced order modes, they do not bring limitations as they do with other methods.
- It uses a flexible criterion and can be tailored to solve different sensor placement goals (see III-C).
- It produces tailored linear estimators. They are designed from the sensor locations and learning sequence only. Prior knowledge on the expected sensor noise can be integrated in the design scheme to make the estimator robust (see III-C).

### III. SENSORSPACE : A STATISTICAL LEARNING SENSOR PLACEMENT ALGORITHM

#### A. Presentation and contributions

The noiseless case is first considered to better illustrate our algorithm. As seen previously, problem (8) is solved by minimizing \(\epsilon\) (7). For a given \(C\), the optimal estimator \(\psi_o\) minimizing \(\epsilon = \|Y - \psi CY\|_F\) is
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This is the estimator used to recover the field of interest. Note that \(\psi_o = \psi_{\text{POD}}\) in cases where \(\epsilon_d = 0\), Eq. (9), and \(n_S \geq n_D\). This does not happen in our framework where \(n_S < n_D\). This condition is more prominent when the sensor locations are constrained as more measurements are needed to overcome the suboptimal positioning and maintain recovery accuracy. With this closed-form expression for \(\psi\), the sensor matrix \(C \in \mathbb{R}^{n_S \times n_x}\) now satisfies
\[
C \in \arg \min_{C \in \mathcal{M}_C} \left\| Y - Y \left( \bar{C}Y \right)^+ \bar{C}Y \right\|_F.
\] (14)

Most solution methods of the literature focus on \(a\) instead of the larger \(y\) since it leads to faster algorithms. However, there is no guarantee that a good sensor placement in terms of recovery of the reduced order modes coefficients \(a\) leads to a good recovery performance of the quantity \(y\) one is really interested in. Through an efficient formulation, the evaluation of the criterion is sufficiently fast to allow different sensor configurations to be tested in a greedy manner. For that purpose, an SVD decomposition \(Y = U\Sigma V^T\) is made. \(n_o\) singular vectors are retained such that less than 0.01% of relative error is made on approximating \(Y\). Note that no consideration on the number of sensors has to be taken into account when choosing \(n_o\), unlike \(n_D\) that clearly impacts the recovery performance. With \(I_{n_o}\), the identity matrix of order \(n_x\), \(U_{n_o}\), the first \(n_o\) columns of \(U\), \(\Sigma_{n_o}\) the restriction of \(\Sigma\) to the first \(n_o\) rows and columns, and \(U_{n_o} = U_{n_o}\Sigma_{n_o}\),
the criterion (7) becomes
\[ \varepsilon_U = \left\| \left( I_{n_x} - Y (CY)^+ C \right) Y \right\|_F, \]
\[ \approx \left\| I_{n_x} - Y (CY)^+ C U_{n_o} \Sigma_n o \right\|_F, \]
\[ = \left\| I_{n_x} - Y (CY)^+ C U_{n_o} \right\|_F, \]
\[ = \left\| \tilde{U}_{n_o} - Y (CY)^+ C \tilde{U}_{n_o} \right\|_F, \]
(15)
since \( V_n^T \) is unitary. Note that \( CY \) and \( C \tilde{U}_{n_o} \) never require direct computation since the specific structure of \( C \) imposes that these matrices are just row restrictions of \( Y \) imposed by the pseudo-inverse of the full row-rank matrix respectively. The numerical complexity associated with computing the pseudo-inverse of the full row-rank matrix \( CY \) is at leading order \( \frac{3}{2} n_S^2 n_t + \frac{1}{3} n_S^3 \) at leading order, [15]. It is cubic in the small number of sensors and only linear in the number of snapshots, hence allowing a fast evaluation. Further, the present SensorSpace method is applied off-line, with no real-time constraint.

C. Goal-Oriented and robust observers

The criterion \( \varepsilon \) can easily be modified for a goal-oriented recovery of \( GY \) instead of \( y \), with \( G \in \mathbb{R}^{n_o \times n_s} \). Selecting the sensors to recover \( y \) and deducing an estimate of the Quantity of Interest \( GY \) is not as effective as using a specific sensor placement and estimator to recover \( GY \) directly. The criterion becomes
\[ \varepsilon_G = \left\| G \tilde{U}_{n_o} - GY (CY)^+ C \tilde{U}_{n_o} \right\|_F. \] (16)

Noisy data is now considered. Each sensor outputs a measurement plagued with additive white noise assumed stationary, statistically independent, and drawn from the same probability distribution. Let \( n_c \) independent realizations of \( \Xi \) be collected in \( \Xi^{ext} \in \mathbb{R}^{n_S \times (n_c n_t)} \) which here bears information on the “training noise”. The sought after \( C \) and \( \psi_n \) should lead to the lowest recovery error if the actual noise is statistically close to the one generating \( \Xi^{ext} \). For a given learning sequence \( Y \), one considers the snapshot matrix \( Y^{ext} = \left[ Y \ldots Y \right] \in \mathbb{R}^{n_S \times (n_c n_t)} \) and measurement matrix \( Y_s^{ext} = CY^{ext} + \Xi^{ext} \) associated to the collection of \( n_c \) noise realizations. This approach allows to map realizations of the noisy measurements to the relevant field of the learning sequence, hence adding robustness to noise to the design process. The robust estimator \( \psi_n \) then minimizes the norm of the recovery error \( \left\| Y^{ext} - \psi_n Y_s^{ext} \right\|_F \), hence
\[ \psi_n = Y^{ext} \left( Y_s^{ext} \right)^+. \] (17)

It is important to note that \( Y_s^{ext} \) is skinny so that its pseudo-inverse is inexpensive to evaluate. Further, \( Y^{ext} \) needs not be stored nor assembled and \( \psi_n \) is simply evaluated with \( n_c \) multiplications of the much smaller \( Y \in \mathbb{R}^{n_S \times n_t} \) matrix to successive \( n_t \) rows of \( Y^{ext} \).

This linear estimator has hence been robustified with respect to noise. Finally, our algorithm can be used simply by modifying the criterion into
\[ \varepsilon_n = \left\| \tilde{U}_{n_o} - \psi_n C \tilde{U}_{n_o} \right\|_F. \] (18)

Once the robust estimator is computed, the noiseless data is used when evaluating the criterion. This is done so that the sensor positions and associated estimators work correctly in the noiseless case.

The pseudo-code can be found in Algorithm 1.

**Algorithm 1 SensorSpace Pseudo-Code**

**Require:** \( Y, n_S, \) pos\( P, n_o \)

1. **Initialization:**
2. \( U \Sigma V^T \leftarrow \text{SVD decomposition of } Y \)
3. \( \tilde{U}_{n_o} \leftarrow U_{n_o} \Sigma_n \)
4. \( \text{posC} \leftarrow \text{initial possible sensor locations} \)
5. \( \text{posC} \leftarrow n_S \text{ distinct random values of pos}\ P \text{ satisfying constraints} \)
6. \( \varepsilon \leftarrow \varepsilon_U \) (or \( \varepsilon_G, \varepsilon_n \))
7. Repeat until the stopping criteria is met:
8. **Main Loop:**
9. Remove a sensor at random from \( \text{posC} \)
10. Update the possible positions \( \text{posP} \) (\( n_p \) elements):
11. \( \text{for } j \in \{1 \ldots n_p \} \text{ do} \)
12. \( C \leftarrow \{ \text{posC}, \text{posP}(j) \} \) locations
13. \( \varepsilon_j \leftarrow \varepsilon_U \) (or \( \varepsilon_G, \varepsilon_n \))
14. Select the sensor in \( \text{posC} \) associated to the smallest \( \varepsilon_j \)
15. **Return:** \( C \leftarrow \text{posC} \) locations

IV. RESULTS

A. Test Environment

To illustrate the performance of our SensorSpace (SS) approach, the numerically simulated flow around a circular cylinder is considered. The flow is laminar, incompressible, two-dimensional and the Reynolds number is 200. Despite this simple configuration, it exhibits many features encountered in more complicated configurations such as boundary layer separation, recirculations and an absolutely unstable region in the near wake. The flow is solved by employing a stream function - vorticity formulation of the 2-D incompressible Navier-Stokes equations. The aim is to command two, independently driven, plasma actuators symmetrically located at the cylinder surface near the boundary layer separation point. The control input is a two-component vector, \( u^T = (u_1 u_2) \in \mathbb{R}^2 \). The control is closed-loop and relies on measures from a few pressure sensors mounted on the surface of the cylinder. The objective is to lower the pressure drag of the cylinder and hence requires the pressure field in a ring-shaped neighborhood of the cylinder for an accurate estimation of the cost functional. To illustrate the configuration, a typical instantaneous relative pressure field is plotted in Fig. 1 where the von Kármán vortices of the wake are clearly visible as low pressure regions. The pressure field enclosed by the dashed line around the cylinder is the field of interest \( y \) and is described by \( n_s = 3015 \) elements. A typical sensor placement (white circles, \( n_S = 5 \)) is shown as well as the two actuators (purple squares).
B. Learning Sequence

The learning sequence $Y$ is now generated. States of the system under typical situations likely to be encountered in practical use are recorded. The necessary command is expected to be directly related to the vortex shedding, hence oscillatory in time and the collection of scenarios then includes constant control actions (including no control at all) as well as commands varying sinusoidally in time at a frequency close to the expected von Kármán frequency. Transition of the system between two control regimes is also an important behavior to capture. The retained learning control sequence is plotted in Fig. 2. Snapshots were taken every 0.4 time unit (non-dimensional) and resulted in $n_t = 1875$.

C. Numerical Results

Sensors can only be set on the surface of the cylinder. This considerably reduces the domain where the sensors can be placed from 3015 to 201 grid points. This situation is hereafter referred to as the unconstrained case. The constrained case is when the sensors cannot be in the vicinity of the actuators and when one takes into account the spatial extension of the sensors by having them be at least 3 grid points away from each other.

Our algorithm is now compared to two popular approaches, EI and FS ([8], [16]). Small modifications were made on both methods to incorporate proximity constraints.

Up to $n_S = 5$ sensors are considered and the goal is to estimate the relative pressure field in the vicinity of the cylinder. Robustness against noise is already included in these two algorithms. Measurement noise, for each sensor and snapshot, consists in independent white noise drawn from a zero-mean Gaussian distribution with a variance of $\sigma_n^2$, $\xi \sim \mathcal{N}(0, \sigma_n^2)$.

Our first test considers the unconstrained noiseless setup, see Fig. 3 (top). The methods are compared in terms of the relative estimation error given by

$$\varepsilon_r = \frac{\|Y - \hat{Y}\|_F}{\|Y\|_F}. \quad (19)$$

All three methods perform very well and lead to a decreasing error when more sensors are available. As expected, our method is comparatively better with few sensors. All methods position some sensors near the actuators. If even more sensors are available, all methods give extremely similar results.

The next setup considered is the constrained noiseless situation, Fig. 3 (bottom). FS is unable to give satisfying results and needs to be modified to incorporate such constraints. Our method clearly outperforms EI. This behavior illustrates the efficiency of using the criteria $\varepsilon$ and associated estimator for constrained sensor placement.

![Fig. 3. Performance in a noiseless environment (top: unconstrained ; bottom: constrained).](image_url)
the solution (sensor placement, estimator) giving the best noiseless ($\sigma_b = 0$) recovery performance is retained. Since this is an off-line step, one can usually afford multiple runs to find the best solution. To compare the performance of EI and SS with respect to noise, the expected value of the relative estimation error $\mathbb{E}[\epsilon_r]$ is computed for different values of $\sigma_n^2$. This expected value is estimated from 100 noise realizations for each $\sigma_n^2$ considered.

For a low training noise, $\sigma_n^2 = 0.01$, SS exhibits good estimation performance but drops when $\sigma_n^2$ increases, eventually under-performing EI’s inherent noise robustness. Using $\sigma_n^2 = 0.05$ is sufficient to give SS noise robustness similar to EI at the cost of losing estimation efficiency in a low noise setting. The proposed approach can hence be made robust by relying on a prior $\mathcal{N}(0, \sigma_n^2)$ on the measurement noise. As expected, the performance is best with the strategy derived with a noise variance $\sigma_n^2$ close to the one indeed encountered $\sigma_n^2$. In contrast, EI does not allow to tailor the solution to a given expected noise level and distribution.

![Performance in a noisy environment (nS = 4).](image)

**V. CONCLUSIONS**

A new sensor placement algorithm (SensorSpace, SS) was presented and its performance was illustrated in a practical case. The sensors are located such that, when used in combination with an optimal, sensors-dependent, linear lift-up operator (also termed “estimator” in the text), they minimize the reconstruction error of a learning sequence of field snapshots. The estimator operator and the sensors location are the joint solution of an optimization problem and are then intrinsically coupled. Our approach allows for superior performance in comparison with other sensor placement techniques from the literature since our previous coupling does not constrain as much the estimator’s structure. Indeed, these other methods typically use an estimator relying on a prior reduced order model of the system at hand. Further, the present SS approach naturally lends itself to account for constraints in the locations of the sensors as well as robustness against noise in the sensors and goal-oriented recovery. In a situation where constraints exist on the sensors, the resulting placement remains optimal when combined with the derived estimator owing to the intricate coupling of the method. This is in contrast with standard approaches such as Effective Independence (EI) or FrameSense (FS) where recovery performance may drop dramatically when constraints are considered since the reduced order basis they rely on is then no longer suitable. While more computationally costly than EI and FS, the SS method is shown to achieve significantly better performance in estimating a bulk field from wall-mounted sensors, in particular when very few sensors are available. Further extension of the present work include a sensor placement methodology for a nonlinear goal-oriented recovery (such as recovering the velocity field from pressure measurements). Superior performance is also achieved when used with a nonlinear, parsimony-aware, estimator and is ongoing work.

**REFERENCES**


