MODELLING NATURAL CONVECTION WITH THE PISTON EFFECT, A THERMODYNAMIC NECESSITY

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ABSTRACT
The usual Boussinesq equations lead to thermodynamic inconsistencies, especially with respect to second law balances. A thermodynamically consistent model, still assuming \( \nabla \cdot v = 0 \), can easily be built that accounts for the work of pressure forces in the heat equation: the piston effect plays a role in buoyancy-induced natural convection, also when the fluid is far from its critical state, also in steady-state.

NOMENCLATURE
\[ A_r \] aspect ratio [-], \[ N_f \] irreversibility numbers [-], \[ q_v \] heat locally generated by viscous friction [W.m\(^{-3}\)], \[ V_z \] vertical component of fluid velocity [m.s\(^{-1}\)], \[ x_c \] position of the cold wall, \( = A_r^{-1} \), [-], \[ \Delta T \] temperature difference \( T_h - T_c \), [K], \[ \phi \] non-dimensional adiabatic temperature gradient

Subscripts
0 reference state, \[ \nu \] viscous, \[ c \] cold side, \[ C \] Carnot, \[ W \] mechanical energy (also \( W_m \)), \[ q \] conductive (heat diffusion), \[ \lambda \] purely conductive system

INTRODUCTION
The famous Oberbeck-Boussinesq equations have been established more than a century ago [1,2], and have been very helpful since then for modeling buoyancy-driven natural convection [3,4]. As long as those equations are used for simulations dedicated to comparison with experimental data, comprehensive studies [5], they surely are pertinent. However, the current studies about natural convection are more and more refined and theoretical, involving second law analyses, stability analyses, or multiple solutions, so that one may wonder whether those equations are still adapted to the intended purposes. It is well-known that the Boussinesq approximation is valid as long as the temperature difference is limited so that the fluid density can be assumed as uniform and constant [6,7]. When this condition is not fulfilled, the problem is said non-Boussinesq; such problems are investigated since the eighties with low-Mach-number models [8]. The concern herein is completely different. Its origin lies in the difference between two entropy balances, on one hand that of real natural convection in steady-state, and on the other hand that of the system simulated with the usual Boussinesq (UB) equations. The consequences of the basic thermodynamic inconsistency of the UB approximation are particularly visible when the temperature difference is very small, i.e. rather close to thermodynamic equilibrium [9]. The present study focuses on such configurations, describes that thermodynamic inconsistency and brings a solution.

NATURAL CONVECTION AND ENTROPY BALANCE
Non-dimensional quantities are well-known for energies, e.g. the Nusselt number is the ratio of the effective heat flux and that transferred by conduction only through the fluid at rest: \( Nu = Q_h/Q_{\lambda} \). In the following, that same quantity \( Q_{\lambda} \) is used for non-dimensionalising any energy rate. The same only-conductive system can also be the reference for entropy balances, so that its entropy production \( \Sigma_{\lambda} \) is used for non-dimensionalising any
rate of entropy production or change. This non-dimensionalisation permits establishing an equality that must exist in steady-state between the number of total irreversibility (\(N_I = \Sigma \lambda \)) and the Nusselt number:

\[
\Sigma = Qh(T_c^{-1} - T_h^{-1}) = Nu \lambda (T_c^{-1} - T_h^{-1}) = Nu \Sigma \lambda
\]

(1)

In thermally-induced natural convection without diffusion (i.e. in pure substances), the sources of irreversibility are heat diffusion by conduction and viscous friction. Non-dimensionalising the corresponding entropy production as above described leads to the following condition:

\[
N_{Iq} + N_{Iv} = Nu
\]

(2)

THERMODYNAMIC INCONSISTENCY IN THE USUAL BOUSSINESQ EQUATIONS

The combination of the Gibbs equation in its local form (\(Tds = du + pdv\)), of the entropy balance \([\rho Ds/Dr = -\nabla \cdot (T^{-1} q) + \sigma]\), and of the UB assumptions, i.e. \(\nabla \cdot \mathbf{v} = 0\), and \((\rho Du/Dr = -\nabla \cdot \mathbf{q})\), results in \(\sigma = q \nabla (T^{-1})\), which means that the only irreversibility recognized by the UB system is the conductive one. Indeed, when calculating \(N_{Iq}\) according to:

\[
N_{Iq} = \frac{1}{A_r} \int_0^x \left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial z} \right)^2 \frac{(1 + \theta \Delta T/T_0)^2}{(1 + \theta \Delta T/T_0)^2} \, dx \, dz
\]

(3)

one always obtains equality of \(N_{Iq}\) with \(Nu\) (where \(Nu = -A_r^{-1} \int_0^1 \left( \frac{\partial \theta}{\partial x} \right)_{x=0} \, dx\)). It can be formally proved that this equality is a consequence of the UB framework. The contradiction with Eq. (2) highlights the thermodynamic inconsistency. In addition, the viscous irreversibility, given by:

\[
N_{Iv} = \frac{1}{A_r} \frac{\beta gH}{c_p} \frac{T_0}{\Delta T} \int_0^x \frac{\Phi}{(1 + \theta \Delta T/T_0)^2}
\]

(4)

where

\[
\Phi = 2 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + 2 \left( \frac{\partial w}{\partial z} \right)^2
\]

displays a parameter, independent from \(A_r\), \(Pr\), and \(Ra\), and which is absent from the UB problem: \(\beta gHT_0/(c_p\Delta T)\). Tritton [7] noticed that this parameter is the adiabatic temperature gradient non-dimensionalised in the problem framework. As this fourth parameter is very often mentioned in the following it will be denoted by the symbol \(\phi\). Bejan [3], and Gebhart et al. [4] as well, mention that \(\phi\) might easily be comparable to one, so that the viscous irreversibility ‘is not necessarily negligible’, as written in both books.

The development about entropy productions at the beginning of this section shows that the usual heat equation of the Boussinesq model (\(DT/Dr = \alpha \nabla^2 T\)) should be questioned. First, it discards the heat generated by viscous friction, while the corresponding loss of kinetic energy is accounted for in the momentum equation. Lavoisier said ‘rien ne se perd’ (nothing disappears) so that, strictly speaking, that kinetic energy (work) which (in the UB system) is lost by viscous friction but not transformed into heat, cannot do anything else than being released as work outside the system (Note that any exact transformation of work into work is a reversible process: again this transfer is not an irreversibility in the UB system). Second, and similarly, that usual heat equation does not contain any term involving transformation between heat (internal energy) and work. In other words, the UB system somehow continuously receives mechanical energy from outside in order to compensate its continuous viscous loss. After this simple analysis, one can schematically represent the energy transfers of the UB system as shown in Figure 1: in addition to the two equal heat fluxes exchanged with the heat sources, this system exchanges two equal and opposite fluxes of mechanical energy with the surrounding, and does not involve any internal transformation of heat into work and vice-versa.
The Thermodynamic Boussinesq Model

In the thermodynamic Boussinesq (TB) model the temperature difference is still assumed small enough for neglecting the changes in fluid density except in the buoyancy term of the momentum equation. However, the heat equation is rewritten from its enthalpic form without discarding any term:

\[
\frac{D T}{D t} = a_0 \nabla^2 T + \frac{q_v}{\rho c_p} + \frac{T}{c_p} \left( \frac{\partial (\rho^{-1})}{\partial T} \right)_p \frac{D P}{D t}.
\]

(5)

This equation accounts for the heat generated by viscous friction and, more important, also for the work of pressure forces. As the present study focuses on steady-states, only the hydrostatic pressure field is considered for the latter term, which thus transforms into: \(-\left(\frac{\beta g}{c_p}\right)TV\). Taking the cavity height \(H\), the speed \(V^*=(\sqrt{Ra} \alpha/H)\), and \(\Delta T\), as respective references for distances, velocity and temperature the non-dimensional form of this heat equation is:

\[
\frac{D \theta}{D \tau} = \frac{1}{Ra^{1/2}} \nabla^2 \theta + \frac{\beta g H}{c_p} \left( \frac{\Phi}{Ra^{1/2}} - \theta_w \right) - \phi_w.
\]

(6)

This equation involves the parameter \(\phi\). It also involves \(\beta g H/c_p\), which is very small (of the order of \(10^{-5}\)), while \(\phi\) is ‘not necessarily negligible’. In addition, when the development about the entropy productions is derived again, but with (5) as heat equation, it is found that the irreversibility has two components, heat diffusion and viscous friction, as it must.

Numerical implementation: The modification of the heat equation is implemented into a usual Boussinesq model, which is described in ref. [10]. The two terms added by this modification are treated like the other non-linear terms, i.e. implicitly through linear extrapolation. The cost of this addition in terms of CPU-time is absolutely negligible. We investigate herein the square two-dimensional differentially heated cavity with adiabatic horizontal walls, and filled with air at 300K \((Pr=0.71)\). The grid (regular staggered) is 256 x 256 (the cases with \(Ra>10^8\) are calculated with 512 x 512). The configurations are stationary \((Ra\text{ ranges from 3000 to }10^8\) ) and \(\phi\) ranges from very small values (10-3, i.e. small cavities where the UB approximation is valid) to large ones (2, for which the largest value of \(H\) is 3.8m).

Thermodynamic balances: all the calculations done with the TB model yield \(Nu_h=Nux\), and \(NWm=NWv\). In addition, the second law balance now agrees with Eq. (2): this model is thermodynamically consistent. The dependence of \(Nu\) on \(\phi\) is shown in Figure 2. It can be seen that the Nusselt number calculated with the TB model is always larger than that in the UB case, quite significantly when \(\phi\) is of the order of unity. When \(\phi\) is small, the relative extent of the difference is \(0.3 \times \phi\) (e.g. 3% for \(\phi=0.1\)).

The piston effect: Looking now at Eq. (6), the work of pressure forces (practically the term \(-\phi w\)) acts as a heat sink there where the fluid flows upward (i.e. close to the hot wall), and as a heat source there where the fluid flows downward (i.e. close to the cold wall). The combination of these two heat sources (positive and negative) acts as a global heat transfer, in parallel to conduction plus advection, by a process called the piston effect [11,12]. When subtracting from the total heat flux that due to conduction plus advection through the vertical mid-plane \((x=0.5)\), the difference is the flux transported by piston effect \((Nu_{Pist})\). The contribution of the piston effect to the total heat flux is shown in Figure 3. It is interesting to note that as soon as convection is developed \((10^4 \leq Ra)\), all the curves practically merge into a single correlation: \(\phi\) is a pertinent parameter. It can also be seen that the piston effect might be quite significant when \(\phi\) is of the order of unity, but also as high as \(1.2 \times \phi\) when \(\phi\) is small, e.g. 12% for \(\phi=0.1\), a value accepted as small in the seventies [6]. First, when considering that current calculations are done in double-precision, one may wonder whether that limit of 0.1 is still pertinent. Second, this finding is important because it reminds that part of the reality of buoyancy-induced natural convection, i.e. the hydrostatic pressure gradient and the work it exerts on the flow, is, not approximated, but simply discarded by the UB equations. As a result, the UB equations discard the piston
effect, although it is an intrinsic ingredient of buoyancy-induced natural convection also in fluids which are far away from their critical points, also in steady-states.
This analysis permits representing the energy transfers occurring in natural convection (and accounted for in the \( TB \) equations), see Figure 4: the internal transformations between heat and work have a double result, a net production of kinetic energy (work) that compensates the loss in viscous friction, and a heat transfer, by piston effect, much more significant than the net produced work.

VALIDITY DOMAINS FOR THE DIFFERENT MODELS
When considering the coordinates \( H \) and \( \Delta T \) in log-log axes (Figure 5), for a given fluid a prescribed value of \( Ra \) corresponds to a line with negative slope (solid lines) while a given value of \( \phi \) corresponds to a line with positive slope (dashed lines). The right-most region (large \( \Delta T \)'s) is that of non-Boussinesq problems and Low-Mach-number models [8]. The open circle there approximately shows configurations simulated in the latter reference with air between 240 and 960K, showing that the non-Boussinesq effects change the Nusselt number by 2-3%. Our study (Figure 2) shows that a comparable change is obtained for \( \phi=0.06-0.09 \), i.e. a \( \Delta T \) of 0.1K applied to a 1m high cavity; this configuration looks quite common.
When the \( \Delta T \) is less than 40-50K, starts the \( UB \) domain. Typical experimental configurations are displayed in Figure 5 (cavity between 10cm and 2m; \( \Delta T \) between 5 and 20K [13]). The corresponding values of \( \phi \) are very weak (\( 10^{-4}-10^{-3} \)), this is why the effects described herein are not observed in experiments. This is also why the \( UB \) model is well-adapted for comparing numerical calculations to experimental data.
After other works [6,7], the present study shows that the \( UB \) is valid only when \( \phi \) is small, thus highlighting a third domain, with small \( \Delta T \)'s and denoted \( TB \) herein. The question, where is the frontier between the \( UB \) and the \( TB \) domains, looks still open. Usually, it was located at \( \phi=0.1 \); the value 0.03 surely is more reasonable, and 0.01 more rigorous. It is interesting to note that configurations that might be found in building engineering (\( \Delta T \approx 1K \) and \( H\approx 3-6m \)) yield \( \phi=0.03 \) (see the bold double-arrow in Figure 5). One may wonder whether the usual Boussinesq model is a reliable basis for investigating optimal-control strategies in such configurations.
A series of four dots can be seen in the \( TB \) domain, for \( H \) slightly less than 4m, and \( \Delta T \) ranging from 0.03 to 0.08K approximately. They are constructed as follows: the case \( Ra=10^8 \) and \( \phi=2 \) is selected; the cavity height is kept constant, and the temperature difference multiplied by 1.5, 2, 3 and finally 4, thus reaching the case \( Ra=4\times10^8 \) and \( \phi=0.5 \).
These four configurations are shown by the four dots of Figure 5. Le Quéré and Behnia [14] have evaluated (with the \( UB \) model) the critical Rayleigh number for onset of unsteadiness in the same configuration as herein and have located it between 1.80\( \times10^8 \) and 1.84\( \times10^8 \). Oppositely, the calculations done with the \( TB \) model for these four cases, i.e. up to \( Ra=4\times10^8 \), all yield a stable flow. It might be necessary to revise some results about stability by including the work of pressure forces and the piston effect in the Boussinesq model.

CONCLUSIONS
The usual Boussinesq equations do not exactly represent buoyancy-induced natural convection. Indeed, the system actually simulated by those equations exchanges two equal and opposite fluxes of mechanical energy with its surrounding. More important, this system recognizes only heat diffusion as irreversibility. It results that the so-simulated entropy balance cannot be what it should be. Thermodynamic consistency is retrieved when, 1/ the work of pressure forces, 2/ the heat generated by viscous friction, are accounted for in the heat equation. This is the thermodynamic Boussinesq model. Numerically, this correction is very simple and negligible in computing time. However, it re-introduces into the system a phenomenon, which is intrinsic to buoyancy-induced natural convection but unfortunately discarded by the usual Boussinesq equations: the piston effect. The piston effect is controlled by the adiabatic temperature gradient non-dimensionalized in the problem framework, \( \phi=\frac{f\rho g H T_0}{(c_p\Delta T)} \), which therefore belongs to the list of control parameters in natural convection. The numerical calculations show that the magnitude of the piston effect can be as large as 1.2\( \phi \), making the thermodynamic Boussinesq model necessary in order to simulate configurations with \( \phi=0.03 \) (maybe 0.01). Moreover, any theoretical study about buoyancy-induced natural convection (second law analyses, very
probably stability analyses as well) should involve the influence of the hydrostatic pressure forces and of the piston effect.

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REFERENCES

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**Figure 1**
Energy diagram describing the usual Boussinesq system in steady-state. Heat fluxes are represented at the lower level, work at the upper level. The exchanges with the surrounding are shown and indicated non-dimensionally (\( Nu_h, Nu_c, NW_m, NW_v \)).

One has \( Nu_h = Nu_c \), and \( NW_m = NW_v \).
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**Figure 2**
Dependence of the Nusselt number on the parameter $\phi$, for different Rayleigh numbers (+: 3000; $\phi$: $10^4$; $\square$: $10^5$; $\bullet$: $10^6$; $\Delta$: $10^7$; $\times$: $10^8$). The Nusselt numbers for $\phi=0$, $Nu_0$, are given in Table 1. The dashed line corresponds to: $(Nu-Nu_0)/Nu_0=0.3\times\phi$.

**Figure 3**
Relative magnitude of heat transfer by piston effect as a function of $\phi$, for different Rayleigh numbers (same convention as in Figure 2). The dashed line corresponds to: $Nu_{Pist}/Nu=1.2\times\phi$.

**Figure 4**
Energy diagram describing natural convection and the thermodynamic Boussinesq system in steady-state (same conventions as in Figure 1). The large triangles symbolize the internal energy transformations between heat and work.

**Figure 5**
Diagram ($H$, $\Delta T$) for representing the different configurations of natural convection. The lines for prescribed values of $Ra$ and $\phi$ are established for air at 300K. The region with large $\Delta T$'s (and -at fixed $Ra$- relatively small cavities and small $\phi$'s) is the one for non-Boussinesq cases. The region with $\phi$ larger than 0.1 (and -at fixed $Ra$- relatively large cavities and small $\Delta T$'s) must be studied with the thermodynamic Boussinesq model. In between is the region of validity of the usual Boussinesq model. The frontiers between the regions are rather diffuse, depending on the purpose of the model. It can be seen that most of the experimental configurations lay in the latter region (see the open ellipse). The thick double-arrow shows that some problems of building engineering are rather close to the thermodynamic region.