On the adequacy of the ten-dimensional model for the wall layer

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In this paper, we provide a numerical validation of the ten-dimensional Proper Orthogonal Decomposition-based model constructed by Aubry et al. [J. Fluid Mech. 192, 115 (1988)] for the wall region of the turbulent boundary layer. Under certain conditions, this model was shown to display intermittent features highly reminiscent of the experimental observations of the bursting process in the wall layer, which makes it a potential key player in understanding and possibly controlling the dynamics of wall-bounded flows. In the same spirit as in our previous study [Podvin and Lumley, J. Fluid Mech. 362, 121 (1998)], we carried out a numerical simulation of a channel flow with relatively small horizontal dimensions which matched those in the 10-D model. The closure hypotheses used to build up the model were confronted with numerical results. Time histories of the modes in the model were compared to those of the simulation. Emphasis was put on identifying long-term characteristics such as a “mean” intermittency period. Our model, quite similar to Aubry’s, was found to display the same heteroclinic cycles under conditions consistent with the numerical experiment. The intermittency period in the model was found to agree well with that found in the simulation. However, the well-ordered character of 10-D bursts is significantly different from the simulation. To try and understand this discrepancy, we simulated a model with streamwise modes (32-D) and found evidence of increasing complexity in the bursts displayed.


I. INTRODUCTION

Over the last 20 years, the theory of dynamical systems has been increasingly used as a tool to investigate problems in fluid mechanics. It provides a mathematical framework which helps analyze and more generally think about the complex phenomena occurring in real flows. The first well-known example of this is Lorenz’s model (1963),1 which emerged from a drastic truncation of the equations for bi-dimensional convection. This model was shown to exhibit chaotic behavior. However, the relationship between abstract mathematical notions such as that of strange attractor and the physical reality they are supposed to describe is far from being clearly established. Definite steps have been made in that direction for closed flows (see, for instance, Ref. 2) such as the Taylor–Couette problem, where the bifurcations occurring in the model appear to correspond to observed modifications in the flow. Generally speaking, one should expect to be able to use relatively low-dimensional systems to represent the behavior of flows involving few spatial scales. The situation is more problematic for turbulent open flows, where no such spatial constraint arises. However, even if a wide variety of scales do interact in these flows, a dominant view in recent years has been that turbulent phenomena are controlled by a small set of relatively well-defined, organized motions—usually referred to in the literature as coherent structures. See, for instance, Cantwell3 or Robinson4 for a review.

One example is the wall region of a turbulent channel flow, i.e., the domain extending from the wall up to a height of about 40 $u_f / \nu$ where $\nu$ is the fluid viscosity and $u_f$ is the friction velocity, which plays a key role in the production and transport of turbulence, both highly intermittent phenomena. Up to relatively high Reynolds numbers, this region has been shown5,6 to be dominated by the presence of streamwise streaks of fluid of alternatively high and low speed. The typical spacing between two low-speed streaks is constant in wall units (i.e., based on the fluid viscosity and friction velocity) and is about 100–150. It is generally assumed that the wall region displays universal features when they are scaled in wall units over a wide range of Reynolds numbers. The streaks are statistically persistent features of the flow. Typically over a finite time they will form, then at some point get lifted up and ejected into the outer flow. There is clear evidence that the instability and break-up of these streaks—also called streak bursting, or bursts—is related to the production of turbulence.7 The violent uplifting of low-speed streaks followed by the gentle downdraft of high-speed fluid near the wall is usually termed the bursting process. These motions can also be seen as strong vortex motions extending from the wall, roughly aligned with the mean flow, hence forming—at least on statistical average—a pair of counter-rotating streamwise vortices. Understanding the dynamics of coherent structures is all the harder since, as Ref. 8 makes it clear, the concept itself of coherent structure precludes the existence of a rigorous definition for it. For instance, up to eight different types of structures have been found in the boundary layer.4

The Proper Orthogonal Decomposition (POD) introduced for turbulence by Lumley (1967)8 stands as a bridge

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between the mathematical, dynamical systems approach and the empirical quest for coherent structures. It provides an objective, energy-based criterion to help identify key motions in the flow. The signal is thus decomposed into an infinity of spatial structures whose amplitude varies in time. The decomposition is optimal in that each structure captures on average as much energy of the signal as possible. In addition, these structures form a complete, orthogonal basis on which the equations of motion can be projected using Galerkin approximation. One is then left with a system of ODE’s for the temporal evolution of the amplitudes of the different structures.

In 1988, Aubry et al. built a five-mode (ten-dimensional) model based on the POD for the wall region of a fully turbulent channel flow. The domain corresponding to the model was able to accommodate in its width two pairs of streaks, or two pairs of streamwise vortices. The model had no streamwise variations, although streamwise variations were implicitly accounted for in the model [see Moffat (1989) and Holmes, Berkooz, and Lumley (1990)]. Since the model represented an extreme truncation of the full flow, the transfer of large-scale energy toward the small scales had to be accounted for. This was done using a Heisenberg-like model, in which the energy transfer was represented by an eddy viscosity characterized by a free parameter \( \alpha \). Strikingly enough, when \( \alpha \) varied within a certain range, this very low-dimensional model displayed intermittent features strongly reminiscent of those of the bursting process observed in the real flow.

This intermittency was shown to be \textit{structurally stable}, i.e., one could show based on symmetry considerations that it would persist in the presence of small perturbations. Variations in turbulent intensity—equivalently the bursting rate—due to physical modifications of the boundary layer, i.e., influence of pressure gradient, shape effect, presence of additional rate of strain or curvature, were shown to be relevant for the undetermined parameter \( \alpha \). The behavior of the POD modes predicted by the model was then compared to that of the modes computed from the simulation.

As a preliminary step, an analogous study was performed for a two-mode (4-D) model representing a domain of very small horizontal dimensions, which Jimenez and Moin have called a Minimal Flow Unit. Such a domain is able to accommodate only a pair of streaks in its width and is therefore highly constrained. The agreement between the behavior predicted by the model and that observed in the simulation, especially in terms of long-term tracking, was surprisingly good, however, such a crude model did not exhibit intermittent features \textit{per se}.

The paper is organized as follows: in Sec. II, we give the details of the numerical simulation. In Sec. III, we briefly describe the Proper Orthogonal Decomposition. We restate how to derive POD-based low-dimensional models in Sec. IV. We then confront numerical integration of the low-D model to results from the direct simulation of the flow. Closure hypotheses are examined in Sec. V. We compare the behavior of the modes in the model and in the simulation in Sec. VI. Conclusion is given in Sec. VII.

\section{II. THE NUMERICAL SIMULATION}

To carry out a systematic validation of the 10-D model computed by Aubry et al., we used the numerical simulation of a fully turbulent channel flow. In what follows, we will refer to the streamwise coordinate as \( x_1 \) or \( x \), the wall-normal coordinate as \( x_2 \) or \( y \), and the spanwise coordinate as \( x_3 \) or \( z \). The velocity field will be indifferently denoted \((u^1,u^2,u^3)\) or \((u,v,w)\). Periodic boundary conditions were applied in the horizontal directions \( x \) and \( z \). Upon examination of the energy spectra, the width of the box was chosen so as to accommodate two pairs of low/high-speed streaks on statistical average. The Reynolds number based on the channel half-height \( h \) and center line velocity of a parabolic profile with the same volume flux \( U_c \) was 4000. The Reynolds number based on the channel half-height and friction velocity \( u_\tau \) was then about 140. The dimensions of the box were \((L_1,L_2,L_3)\) = \((4 \pi/3, 2 \pi/3, 2 \pi/3)\) outer units or \((600,280,300)\) wall units. Wall units will be denoted + throughout the paper. There are no explicit streamwise variations in the 10-D model, so we picked a streamwise dimension roughly corresponding to the length of an individual streak.

We used a pseudo-spectral Navier–Stokes integrator kindly provided by Peter Blossey. Simulations were run on a Cray C98 at IDRIS-CNRS research facilities. The mass flow rate was kept constant. The representation of the flow was based on Fourier modes for the horizontal directions \((x,z)\) and Chebyshev polynomials for the wall-normal direction \( y \). Nonlinear terms were advanced explicitly in time using a third-order Runge–Kutta method in conjunction with a Crank–Nicolson implicit second-order scheme for viscous terms. We used 65 Chebyshev modes in the wall-normal direction, and, respectively, 96 and 48 Fourier modes (before dealiasing) in the streamwise and spanwise directions. Adequate resolution was confirmed by examination of the spectra. As Kravchenko and Moin’s results have shown, dealiasing using the 3/2 rule in the horizontal directions turned out to be quite instrumental in maintaining numerical accuracy. Previous runs carried out in the same box without dealiasing led to decaying turbulence.
The turbulent flow was obtained by superposing a perturbation on a laminar profile and integrating in time. In order to save computation time, we first let the flow develop in a box half its final streamwise and spanwise lengths—in effect a Minimal Flow Unit. When the flow became statistically stationary, we doubled the length and width of the domain by piecing four of these boxes together. A small perturbation was then introduced in the new box and the flow was again allowed to develop. The dimensions of the box remain relatively small compared to those used for a full flow: We are only simulating a “double” Minimal Flow Unit.

We recall that a Minimal Flow Unit was defined by Jimenez and Moin as the smallest physical domain able to sustain turbulence. They found a critical spanwise dimension of about 100 wall units below which turbulence decayed. It seemed that the flow had to be able to contain one pair of high/low-speed streaks in its width to remain turbulent. The Minimal Flow Unit therefore consisted in an infinite array of the same basic cell containing one single coherent structure. Low-order turbulent statistics were in good agreement with experimental observations at least in the wall region. However, some discrepancy, probably reflecting the fact that the dynamics of a unique structure are represented, could be observed in the outer region. Figures 1 and 2, respectively, compare the mean velocity profile and turbulent intensities for the Minimal Flow Unit, the present box ("Double" Minimal Flow Unit) with Kim, Moin, and Moser’s statistics for a full channel. Interestingly, allowing two structures in the box improved the statistics and smeared out most of the discrepancy. Note that the channel half-height in wall units $h^*/u_* = 140$ for our box is slightly different from that for both the full flow unit and the Minimal Flow Unit $(180)$, which may make comparison difficult in the core region of the channel.

A typical feature of the Minimal Flow Unit was the presence of intermittency in all turbulent statistics obtained by spatial averaging. The spatially averaged wall drag typically varied by about 30% over a characteristic period of $O(100)$ outer units $h/U$. As Fig. 3 shows, this is also the case in the Double Minimal Flow Unit. Drag excursions of about 20%–30% occur over time scales of 100–200 outer units.

In the next two sections we essentially reproduce material from our previous paper.17 The reader is referred to the work of Aubry et al.,10 or to the review of Berkooz et al.,22 or to the book by Holmes, Lumley, and Berkooz.23

III. THE PROPER ORTHOGONAL DECOMPOSITION

Let us consider the velocity field in the channel $\vec{u}(x,y,z,t)$. This field can be decomposed into a mean part $\overline{U}(y,t)$ (the average is performed over horizontal planes $y = cst$) and a fluctuating part $\overline{u}(x,y,z,t)$:

$$\overline{u}(x,y,z,t) = \overline{U}(y,t) + \overline{u}(x,y,z,t).$$

![FIG. 1. Mean velocity profile. Data for full flow unit courtesy of R. Moser.](image1)

![FIG. 2. Turbulent intensities: --- u; -- v; - - - w.](image2)

![FIG. 3. Spatially averaged drag history at the bottom wall in the DNS.](image3)
The Karhunen–Loève theorem states that there exists an infinite set of eigenfunctions and eigenvalues \((\phi^n, \lambda^n)\) such that

\[
u(x,y,z,t) = \sum n a^n(t) \phi^n(x,y,z),
\]

where \(a^n\) are uncorrelated random coefficients of variance \(\lambda^n\). These eigenmodes correspond to the structures which are best correlated with the fluctuating velocity field. In statistically homogeneous directions, in our case horizontal directions, POD eigenmodes are simply Fourier modes. The problem can then be carried into Fourier space and one finds that the eigenfunctions \(\phi^n\) associated with the eigenvalue \(\lambda^n\) are solutions of

\[
\int \langle \hat{u}^n_k(y) \hat{u}^n_k(y') \rangle \phi^n_k(y) dy' = \lambda^n \phi^n_k(y),
\]

where \(\langle \cdot \rangle\) denotes an ensemble average, \(\hat{u}_k(y,t)\) represents the Fourier transform of the velocity field in the horizontal directions and \(\Phi^i(x,y',z') = \langle \hat{u}_k(x,y,z,t) \hat{u}_l(x',y',z',t) \rangle\) is the Fourier transform of the spatial autocorrelation tensor at zero time lag.

Let \(k = (k_1, k_3)\). The velocity field is then decomposed as follows:

\[
\hat{u}_{k_1,k_3}(y,t) = \sum_{k_1,k_3} a^n_{k_1,k_3}(t) \phi^n_{k_1,k_3}(y).
\]

We choose to normalize the eigenfunctions such that

\[
\int_{\text{domain}} \phi^n_{k_1,k_3}(y) \phi^n_{k_1,k_3}(y) dy = 1,
\]

so that

\[
\langle a^n_{k_1,k_3}(t) a^n_{k_1,k_3}(t) \rangle = \lambda^n_{k_1,k_3}.
\]

The decomposition is optimal in the sense that the first \(n\) POD eigenfunctions capture more energy on average than any other decomposing set of \(n\) elements. Assuming zero-phase lag between consecutive wave numbers, the first structure \((n = 1)\) for the wall layer turns out to represent a pair of counter-rotating streamwise vortices lifting low-speed streaks of fluid between them and pushing downward high-speed fluid. The first structure and the first three structures, respectively, account for 60% and 90% of the total kinetic energy of the wall region.

We applied the POD to our numerical simulation by constructing the autocorrelation tensor restricted to the wall layer \(0 \leq y \leq 55\) wall units and solving Eq. (3). The procedure was strictly identical to that described in Ref. 17. We quadrupled our initial number of samples by making use of the kernel symmetries

\[
\Phi_{k_1,k_3}^{ij}(y,y') = \Phi_{k_1,k_3}^{ij}(y',y),
\]

\[
\Phi_{k_1,k_3}^{ij}(y,y') = \omega_{ij} \Phi_{k_1-k_3}^{ij}(y,y'),
\]

where

\[
\omega_{ij} = \begin{cases} 1 & \text{if } i \text{ or } j \text{ are both equal to or both} \\ -1 & \text{different from } 3, \\ 0 & \text{otherwise.} \end{cases}
\]

Table I shows the eigenvalue spectrum for the zero and first nonzero streamwise mode \(\lambda^1_{k_1,k_3}\) for \(k_1 = \{0, 1\}\). As was found in Ref. 17, the eigenvalues, at least for the first-order structures, were found to be quite similar to those of a real flow. The maximum of the Reynolds stress was reached at about 30 wall units. As expected, the most energetic mode was the second spanwise mode since the box will accommodate on average two pairs of streaks.

### IV. POD-BASED LOW-DIMENSIONAL MODELS

Let us introduce a few useful notations. In practice, we consider a finite set of POD modes. Let

\[
T(l,m,n) = \{a^n_k | k_1 = l, k_3 = m, p = n\},
\]

so that we can write

\[
u' = u'_< + u'_>,
\]

where \(u'_<\) represents the projection on the space spanned by \(T(l,m,n)\) and \(u'_>\) represents the complement of that projection into the unresolved space.

We briefly outline how the model equations are obtained. Starting from the Navier–Stokes for an incompressible fluid

\[
\frac{\partial \hat{u}}{\partial t} = - \hat{u} \frac{\partial \hat{u}}{\partial x} - \frac{1}{\rho} \frac{\partial \hat{p}}{\partial x} + \nu \frac{\partial^2 \hat{u}}{\partial x^2},
\]

and introducing the decomposition Eq. (1) into them, we obtain

\[
\frac{\partial u'}{\partial t} = - u' \frac{\partial u'}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u'}{\partial x^2}(N').
\]

To obtain an evolution equation for the mode \(a^n_k\), we project \(N'\) onto the corresponding eigenfunction, that is we take its Fourier transform \(N'_k\) then compute the inner product

\[
\int_{\text{domain}} N'_k \phi^n_k.\]

For any finite truncation of POD modes \(T(l,m,n)\), the set of ODE’s obtained is not in closed form. Modeling assumptions are required in order to close the system.
A. The mean velocity profile

We first need a correct expression for the mean shear, where “mean” here refers to a horizontal spatial average which we will denote from now on by $\langle \rangle$. The mean velocity profile $\langle \vec{u} \rangle = U(y,t)$ is a function of the vertical position, and also varies in time. Since the mean shear represents the main source of energy for the turbulence, some feedback representing the extraction of energy by the large scales should be incorporated in the expression for the mean velocity profile, as opposed to an infinite energy supply. The derivation of the expression can be found in Ref. 10. From the equation for the mean velocity profile $\vec{U} = U(y,t) \vec{e}_2$ we obtain the following model:

$$\frac{\partial \vec{U}}{\partial t} = -\frac{\partial \langle \vec{uv} \rangle}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial \vec{U}}{\partial y^2},$$

we obtain the following model:

$$U(y,t) = \frac{u_z^2}{\nu} \left( y - \frac{y^2}{2h} \right) + \frac{1}{\nu} \int_0^y \langle \vec{uv} \rangle(y',t)dy'.$$

The expression is exact if the mean velocity field is stationary. Berkooz et al.24 showed that provided that the box dimensions are sufficiently large, the temporal variations of the mean velocity field can be expected to be slow compared to the time scale of the model.

To put the expression in closed form, we represent $\langle \vec{uv} \rangle$ by its projection on the space spanned by the truncation $T(l,m,n)$. This gives cubic terms which are all strictly negative when there are no streamwise variations and only one eigenfunction is considered.

B. The small-scale stress tensor

At the other end of the spectrum, energy is transferred from the larger resolved to the smaller unresolved scales, which defines a cascade process. In practice, we usually consider severely restricted truncations so that we need to provide some mechanism by which energy from the large scales may be dissipated. We assume that large scales are largely insensitive to the details of energy dissipation. Consequently, a simple eddy viscosity transport model or Heisenberg model was employed.

We make the following decomposition:

$$u'u'i = u'_x u'_x + \left[ u'_y u'_x + u'_x u'_y + u'_y u'_y \right].$$

The second and third terms are called Leonard stresses. They represent cross-interactions between resolved and unresolved modes. They have been found to be relatively unimportant25 and were neglected in Aubry’s derivation. Nevertheless for accuracy’s sake, we included them in the expression for the small-scale stress tensor. Let

$$-\tau_{ij} = u'_i u'_j - u'_x u'_x.$$  

We define

$$\tau_{ij} = \langle \tau_{ij} \rangle - \langle \tau_{ij} \rangle - \frac{1}{2} \delta_{ij} \langle \tau_{kk} \rangle.$$  

where $\langle \rangle$ is an operator averaging over the unresolved scales and $\delta_{ij}$ is the Kronecker delta. The eddy viscosity model stipulates that the anisotropic small-scale stress tensor must be proportional to the strain rate of the resolved scales $s_{ij}$:

$$\tau_{ij} = 2 \nu_T s_{ij} = \nu_T (u'_i u'_j + u'_j u'_i),$$

where the commas stand for differentiation. Let

$$\nu_T = \alpha \nu_T = \alpha u \lambda,$$

where $u \lambda$ and $\lambda \alpha$ are, respectively, velocity and length scales characteristic of the unresolved modes, and $\alpha$ is a nondimensional parameter $O(1)$ which characterizes the amount of energy being dissipated. $\alpha$ is expected to vary in a real flow, and different values of $\alpha$ may correspond to different dynamical behaviors. In our simulation, we found $\nu_T/\nu = 2.76$.

The Navier–Stokes equations become

$$
\frac{\partial \vec{u}_k}{\partial t} = -\frac{\partial \vec{u}_k}{\partial x_j} - \frac{U}{j} \frac{\partial \vec{u}_l}{\partial x_j} - \frac{U}{j} \frac{\partial \vec{u}_l}{\partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \left( p + \frac{1}{3} \frac{\partial \vec{u}_k}{\partial x_k} \right) + \nu \frac{\partial \vec{u}_k}{\partial x_k} + \alpha \frac{\partial \vec{u}_k}{\partial x_k} (N \alpha),
$$

The treatment of the pressure term and the isotropic part of the small-scale stress tensor, i.e., the pseudo-pressure term $1/3 \langle \vec{u}_k^2 \rangle$, is briefly mentioned in the Appendix and explained at length in Ref. 25.

C. Higher-order wall-normal eigenmodes

We now consider the POD higher modes $a^\xi_k \in H(l,m,n)$ where

$$H(l,m,n) = \{ a^\xi_k | k_1 < l, k_3 < m, p > n \}.$$ 

Such modes make a contribution to the linear term (see, for instance, the Appendix). Owing to the huge energy gap between consecutive eigenmodes, they were neglected by Aubry et al. Higher-order modes from $H(l,m,n)$ also appear in the unresolved part of the stress tensor $u'u'$. The contribution from the unresolved modes in $H(l,m,n)$ can be included with either the resolved quadratic terms or the small-scale tensor containing high Fourier modes, which we will denote from now on by

$$\tilde{\tau}^{\xi}_{ij} = \langle \tilde{\tau}_{ij} \rangle - \langle \tilde{\tau}_{ij} \rangle - \frac{1}{2} \delta_{ij} \langle \tilde{\tau}_{kk} \rangle.$$  

where $\langle \rangle$ is an operator averaging over the unresolved scales and $\delta_{ij}$ is the Kronecker delta. The eddy viscosity model stipulates that the anisotropic small-scale stress tensor must be proportional to the strain rate of the resolved scales $s_{ij}$:

$$\tau_{ij} = 2 \nu_T s_{ij} = \nu_T (u'_i u'_j + u'_j u'_i),$$

where the commas stand for differentiation. Let

$$\nu_T = \alpha \nu_T = \alpha u \lambda,$$

where $u \lambda$ and $\lambda \alpha$ are, respectively, velocity and length scales characteristic of the unresolved modes, and $\alpha$ is a nondimensional parameter $O(1)$ which characterizes the amount of energy being dissipated. $\alpha$ is expected to vary in a real flow, and different values of $\alpha$ may correspond to different dynamical behaviors. In our simulation, we found $\nu_T/\nu = 2.76$.

The Navier–Stokes equations become

$$
\frac{\partial \vec{u}_k}{\partial t} = -\frac{\partial \vec{u}_k}{\partial x_j} - \frac{U}{j} \frac{\partial \vec{u}_l}{\partial x_j} - \frac{U}{j} \frac{\partial \vec{u}_l}{\partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \left( p + \frac{1}{3} \frac{\partial \vec{u}_k}{\partial x_k} \right) + \nu \frac{\partial \vec{u}_k}{\partial x_k} + \alpha \frac{\partial \vec{u}_k}{\partial x_k} (N \alpha),
$$

The treatment of the pressure term and the isotropic part of the small-scale stress tensor, i.e., the pseudo-pressure term $1/3 \langle \vec{u}_k^2 \rangle$, is briefly mentioned in the Appendix and explained at length in Ref. 25.
V. VALIDATION OF THE CLOSURE HYPOTHESES

From now on, we write $u_\omega$ for the Fourier transform of the velocity $u_k$. Following Aubry et al.’s approach, we select the five most energetic modes, i.e., the first five spanwise Fourier modes with no streamwise variation corresponding to the first wall-normal eigenmode and build a ten-dimensional model (each mode being complex). Since we only consider the first-order structure, we write $a_k$ or equivalently $a(0,k)$ for $a_{0k}$ whenever possible. In all that follows, the POD modes from the model or the simulation are normalized by the square root of their eigenvalue.

The model coefficients were computed using the eigenfunctions obtained by applying the POD to the numerical simulation. The sign and magnitude of the coefficients was quite similar to those in Aubry’s study. The only free parameter in the model is the constant $\alpha$ which regulates the amount of energy transfer to unresolved scales. The value of $\alpha$ will be determined based on results from the numerical simulation of the domain. In the numerical simulation, all terms can be computed in closed form and then confronted to their representation in the low-D model. To assess the validity of the closure assumptions, we rely on low-order statistics such as correlation coefficients—denoted $C$—as well as linear stochastic estimation (LSE) (see, for instance, Ref. 26). To this effect we introduce the notation

$$ (a|b) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \text{Re}[a(t)b^*(t)] dt, $$

(11)

and

$$ \|a\|_1 = (a|a). $$

(12)

We show results for the bottom wall of the channel. Results for the top wall were found to exhibit the same general trends.

A. The mean velocity profile

In the exact equation for mode $a_{0k}$, the contribution of the mean velocity profile is

$$ L(0,k) = \int_{\text{wall layer}} -Uzv_{0k}\phi_{0k}^1 dy. $$

(13)

In the 10-D model, we have seen that only the interaction of the mean with the first eigenmode is taken into account:

$$ L(0,k) = \int_{\text{wall layer}} -Uz a_{0k} \phi_{0k}^2 \phi_{0k}^1 dy. $$

(14)

The mean velocity is modeled as indicated in Sec. IV, giving rise to linear terms $b_k^1 a_{0k}^1$ where

$$ b_k^1 = \int_{\text{wall layer}} u_k^2 / \nu (1-y/h) \phi_{0k}^2 \phi_{0k}^1 dy, $$

(15)

and cubic terms $d_{k,k'}^1 a_{0k}^1 a_{0k'}^1 a_{0k''}^1$ where

$$ d_{k,k'}^1 = \int_{\text{wall layer}} \frac{1}{\nu} \phi_{0k}^1 \phi_{0k'}^1 \phi_{0k''}^1 \phi_{0k}^1 dy, $$

(16)

which represent the feedback part of the mean velocity interaction

$$ F_k = -\frac{1}{\nu} \int_{\text{wall layer}} \langle uv \rangle \phi_{0k}^2 \phi_{0k}^1 dy. $$

(17)

As in Aubry’s study, the linear coefficients representing interaction with the first term in Eq. (6) were real and positive. The cubic coefficients due to the second term in Eq. (6), feedback term, were all real and negative, and dominated by the contribution of the second mode.

We compared $L(0,k)$ and $L(1,k)$. Their magnitude was similar and their correlation very good—about 0.7–0.8—suggesting that the influence of the higher-order structures can indeed be neglected as Fig. 4 shows. A relevant quantity to estimate is the amount of energy supplied on average by these terms; in other words the value of the coefficient if we assume the mean velocity profile to be constant. This can be obtained from LSE by looking for $b_k^1$ such that $\|L(0,k)(L(1,k))-b_k^1(a_0^1)^2\|^2$ is minimum. This yields

$$ b_k^1 \approx \frac{(L(0,k)(L(1,k))^1(a_0^1)(a_0^1)^1)}{(a_0^1)(a_0^1)} $$

(18)

For all modes, we found that $b_k^1$ was positive and represented about 20% of the coefficient $b_k^1$.

We now examine the model used for $U$ itself, by comparing the expressions for the total feedback

$$ F_k = -\frac{1}{\nu} \int_{\text{wall layer}} \langle uv \rangle \phi_{0k}^2 \phi_{0k}^1 dy $$

(19)

with

$$ \int_{\text{wall layer}} -Uz \phi_{0k}^2 \phi_{0k}^1 dy. $$

(20)

The correlation between these two terms was found to be excellent—about 0.7–0.8—for all modes.

Computing the time average of $\int_{\text{wall layer}} -(Uz-(1/\nu) \times \langle u v \rangle) \phi_{0k}^2 \phi_{0k}^1 dy$ yielded out $b_k^1$ to within a few percent.
The quasi-stationarity hypothesis derived in the boundary layer approximation is therefore a good representation of reality. At each instant the mean velocity profile can be assumed to be in quasi-equilibrium.

What now remains to be examined is how the truncated feedback used in the model compares with the full expression \( F_k \). The five-mode truncation captures on average only about 50% of the total Reynolds stress so we expect to underestimate the feedback term; recall that as the Reynolds stress grows, we assume that less energy is supplied to the large scales—and therefore overestimate the contribution of the mean velocity profile. That did turn out to be the case. On average the model feedback represents only 60% of the total Reynolds stress so we expect to underestimate the feedback term; recall that as the Reynolds stress grows, we assume that less energy is supplied to the large scales—and therefore overestimate the contribution of the mean velocity profile. That did turn out to be the case.

In conclusion, the closure hypothesis used to derive the mean velocity profile, which offers the considerable benefit of making the model globally stable by preventing the structures to grow indefinitely, has proved to be a fully adequate mean velocity profile. That did turn out to be the case.

The correlation between the terms as they appear in the equations, i.e., when they are factored with \( a_k \) remains extremely good, about 0.9. This again supports the idea that the average effect of the small scales should be properly accounted for, but that the details of their interaction are relatively unimportant.

In conclusion, the closure hypothesis used to derive the mean velocity profile, which offers the considerable benefit of making the model globally stable by preventing the structures to grow indefinitely, has proved to be a fully adequate picture of reality. However, the crudeness of the truncation considered here leads to an overestimation of the source term. This means that to make our model more realistic, we will have to compensate for this effect by providing extra dissipation (see below).

**B. Higher-order eigenmodes**

In the model, we consider quadratic interactions between terms in the truncation (see Appendix)

\[
Q^M(0,k) = \sum_{(k',k-k') \in \text{truncation}} C^{111}_{k'k-k} a^1_{0k-k} a^1_{0k'},
\]

(21)

In the real equations, higher-order modes interact as well:

\[
Q^T(0,k) = \sum_{(k',k-k') \in \text{truncation}} \int u_{0k'}^i \frac{\partial u_{0k-k'}^{0j}}{\partial x_j} \phi_{0k}^{0i} \, dy
\]

\[
= \sum_{(k',k-k') \in \text{truncation}} C^{pq1}_{k'k-k} a^p_{0k-k'} a^{0q}_{0k'},
\]

(23)

The correlation between \( Q^M \) and \( Q^T \) was found to be very good, typically 0.8. However, this does not necessarily imply (see for instance Fig. 7) that the effect of the higher eigenmodes is totally negligible. Table II shows that the best linear fit of \( Q^T \) to \( Q^M \) is close to 1 for the last three modes, but closer to 1.5 for the most energetic (i.e., first two) modes. As the next two lines in the table show, these modes mainly...
provide energy to higher modes through quadratic interaction, while the last two mainly extract energy. These higher coefficients mean that the first two modes give a nonnegligible amount of their energy to higher-order eigenmodes. In fact, this represents energy transfer to wall-normal smaller scales. As noted before, one can include this contribution with that to horizontal (Fourier) smaller scales, which we now look into.

C. Energy transfer to small scales

In the real equations for $a_k$, energy is exchanged with small scales

$$T(0,k) = \sum_{\{(l,k),(l,k-k')\} \text{ a truncation}} \int u_{\text{tr}}^j \frac{\partial u_{\text{tr}}^j}{\partial x_i} \phi_{\text{tr}}^{ij} dy.$$  \hspace{1cm} (24)

This term is modeled using a Heisenberg-like turbulent viscosity hypothesis (see Sec. IV), so that it is assumed proportional to $-a_k$:

$$T^{\text{H}}(0,k) = -\alpha \nu_T b_k^2 a_k.$$  \hspace{1cm} (25)

Table III shows the correlation between $T(0,k)$ and $a_k$; see also Fig. 8. It rather supports the closure hypothesis. Adding the effect of higher-order eigenmodes, i.e., considering $T^{\text{Q}}(0,k) = T(0,k) + Q^1(0,k) - Q^{\text{H}}(0,k)$ slightly improved the correlations.

To compute a characteristic value for the energy transfer, we define $\alpha$ to minimize

$$\sum_{k=1}^5 \lambda_k \|T(0,k) - \alpha \nu_T b_k^2 a_k\|^2.$$  \hspace{1cm} (26)

We rescale each term with $\lambda_k$ to compensate for the fact that the equations have been normalized. This problem is solved at each instant, so that $\alpha$ varies in time:

$$\alpha = \frac{1}{\nu_T} \frac{\sum_{k=1}^5 \lambda_k \text{Re}[T(0,k) b_k^2 a_k^2]}{\sum_{k=1}^5 \lambda_k b_k^2 a_k^2}.$$  \hspace{1cm} (27)

Figure 9 shows the temporal evolution of $\alpha$ and $\alpha_Q$ which, respectively, correspond to $T^{\text{Q}}(0,k)$ and $T(0,k)$. Clearly higher-order eigenmodes, i.e., modes in $H(0,1.5)$ account for only a small part of the energy transfer. On average $\alpha_Q$ is higher than $\alpha$ by less than 20%. As expected from cascade theory, $\alpha_Q$ is always positive. Note that it is a global effect. When computing $\alpha$ or $\alpha_Q$ for single modes that is minimizing individual terms in the sum we typically encountered strong negative as well as strong positive values.

It is also important to realize that we are considering the intermittency of the entire domain and not isolated events such as those detected by a probe in an experiment. We therefore used a low-pass filter to average over individual ejections—in effect a moving average of about 140 wall units, which represents the time that it would take a blob of fluid originating near the wall to reach the center of the channel. Figure 10 shows that the temporal evolution of $\alpha$ and $\alpha_Q$ is rather well-correlated (0.5) with the drag history; the correlation is still 0.45 without any averaging. This seems to indicate that the amount of energy transferred to the small scales from the streamwise invariant modes of the truncation determines to some relatively large extent the intensity of the

<table>
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<tr>
<th>$k$</th>
<th>$k=1$</th>
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<th>$k=3$</th>
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<th>$k=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^T(0,k)$</td>
<td>$Q^H(0,k)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C(T(0,k), a_k)$</td>
<td>$-0.49$</td>
<td>$-0.79$</td>
<td>$-0.95$</td>
<td>$-0.76$</td>
<td>$-0.77$</td>
</tr>
<tr>
<td>$C(T(0,k) + Q^1(0,k) - Q^H(0,k), a_k)$</td>
<td>$-0.64$</td>
<td>$-0.80$</td>
<td>$-0.78$</td>
<td>$-0.68$</td>
<td>$-0.66$</td>
</tr>
</tbody>
</table>

TABLE III. Statistics for energy transfer to unresolved scales.
burst. This is also what we found in Ref. 17 for the minimal flow unit.

According to cascade theory, a majority of the transfer goes to the next higher scales. We therefore define two analogous versions of $a$, $a_S$ and $a_1$ which are based on energy transfers respectively restricted to all eigenmodes of the first streamwise mode, i.e., $a_{lk}$ with $(|l| = 1, |k| \leq 5)$ and to the first eigenmode of the first streamwise mode, i.e., $a_{1k}$ with $(|l| = 1, |k| \leq 5)$. Histories of $a_S$ and $a_1$ are shown in Fig. 10 and illustrate the complexity of the energy transfer mechanism. It appears impossible to predict the behavior of the drag history based on the simple knowledge of $a_1$. On average $a_{VT}$ was about 2.00. $a_S$ represented more than 50% of the average transfer and $a_1$ only 15%.

D. Model dissipation

We have examined the closure assumptions made in deriving the model, i.e., the influence of the unresolved scales on the modes of our truncation. Our goal is now to make the model as realistic as possible by adjusting the dissipation parameter $\alpha$. We thus define

$$A(0,k) = L(0,k) + Q(0,k) + T(0,k) - (b_k^1 + d_{kk} |a_k|^2) a_k - Q^M(0,k).$$

(28)

$A(0,k)$ represents the difference between the full terms and the explicit, resolved terms in the model—modulo neglected terms (see the Appendix). Keeping true to the intuition that inspired these models, we represent the effect of the unresolved scales as extra dissipation. We therefore look for $\alpha_T$ so that

$$\sum_{k=1}^{5} \lambda_k \left[ A(0,k) - \alpha_T v_f b_k^2 a_k \right]^2$$

is minimum. We found that $\alpha_T v_f$ was about 7.0 on average or equivalently $\alpha_T = 2.50$. Figure 11 shows that the quadratic interactions account for only a small part of the total dissipation introduced in the model. Clearly it is important to make a correct estimation of the energy contributed by the mean shear. As comparison of Fig. 11 and Fig. 12 shows, $\alpha_T$ is not well correlated with the wall drag history, which is no real surprise. $\alpha_T$ mostly accounts for the unresolved scales of the feedback term. The feedback term essentially represents

FIG. 10. Low-pass filtered histories of characteristic indicators in the DNS: --: spatially averaged wall drag history; ---: energy transfer to unresolved modes $a_{VT}$; --: $a_{VT}$ represents the energy transfer to first streamwise modes $a_{ik}$ with $n > 1$; ...: $a_{VT}$ represents energy transfer to first streamwise modes $a_{1k}$.

FIG. 11. Characteristic value for extra dissipation added to the model. $\alpha_T v_f$ represents the total dissipation; $\alpha_T v_f$ represents the dissipation added to compensate for the overestimated source term (underestimated mean velocity feedback).

FIG. 12. Spatially averaged wall drag history in the DNS. The horizontal line represents the temporal and spatial average.
the Reynolds stress in the outer portion, which tends to be anti-correlated with the wall drag, a consequence of mass conservation. The modeled feedback is only partially negatively correlated with the wall drag, so that the difference between the two may not be correlated one way or another.

VI. INTERMITTENCY

A. The 10-D model

In this section we compare the predictions of the 10-D model to the behavior of the true POD modes. These are obtained by taking the inner product of the velocity field with the corresponding eigenfunction

\[ a^{\text{true}}_k = \frac{1}{(\lambda_k^2)^{1/2}} \int_{\text{wall layer}} u_i \phi_i^{n_{\text{true}}} \, dy. \]  

Figure 13 shows the intensity of the modes \( a_k \). The intermittency time scale \( O(100) \frac{h}{U} \) is apparent even if the way the intensity of the modes relates to the wall drag history remains unclear. We only found some evidence that on the one hand, the intensity of the second mode is partly related to the wall drag excursions, and that on the other hand the amplitudes of the first two modes are negatively correlated—about \(-0.4\) for the bottom wall and \(-0.6\) for the top wall.

The main question is whether the 10-D model exhibits intermittency for a value of \( \alpha \) consistent with the numerical simulation. In the last section we found from the DNS a value for \( \alpha \) of 2.50. A rapid study showed that the model displays stable heteroclinic cycles similar to that found by Aubry et al. in the range \( \alpha \in [1.6, 2.95] \). Although a formal proof of their existence is still lacking, 27 there is ample numerical evidence of these cycles. One such cycle consists in two heteroclinic connections between two fixed points sitting diametrically across the torus of equilibria \((|a_2| = \text{cst}, 2 \arg[a_2] - \arg[a_4] = \pi)\) in the even subspace. Each equilibrium is a saddle point with a one-dimensional unstable manifold (not two-dimensional, as was the case in Aubry’s study) in the odd subspace \((a_1, a_3, a_5)\) which intersects the stable manifold of the opposite fixed point on the torus.

When no noise is present in the equations, starting from an initial condition in the basin of attraction of the cycle, the system spends periods of continually increasing length near the equilibria, although in practice the period will reach a finite limit as computer round-off error prevents the system from getting infinitely close to the equilibria. However, we expect noise to be present in the equations: The pressure term acts as a forcing perturbation representing the effect of the outer layer at the top of the domain. Maybe more importantly, the difference between the terms containing unresolved scales in the real equations and their modeled representation in the 10-D system constitutes another source of perturbation. The effect of noise is to maintain the system at some small—finite—distance from the heteroclinic connection \( s \), thereby imposing a “statistical” intermittency period. For sufficiently high noise amplitudes, the system may be pushed along the torus of fixed points, giving rise to traveling heteroclinic cycles.

To obtain an order of magnitude of the bursting period, we added Gaussian perturbations to the equations. We did several tests using characteristic time scales between 1 and 10 outer units, with noise levels in the range \([0.05, 0.30]\). Figure 14 shows the results of the numerical integration for a noise level of 0.05 with a characteristic time scale of 10 outer units. In another test we set the noise level to equal the variance of \( A(0, k) - \alpha T \frac{\partial^2}{\partial t^2} a_k \) (which yielded values comprised between 0.05 and 0.25). In all cases the intermittency period was about 200 outer units, in close agreement with the intermittency period exhibited by the turbulent indicators in the DNS. As Jimenez and Moin noted, the intermittency

FIG. 13. Magnitude of the POD coefficients in the DNS: \(|a_k|, k = 1\) to 5 from bottom to top.

FIG. 14. 10-D model with noise: \( \text{Re}[a_k], k = 1\) to 5 from bottom to top.
period "is a very long time scale, difficult to reconcile with any obvious property of the flow or of the simulation." They showed in particular that this time scale did not depend on the streamwise dimension of the box. This is in agreement with the streamwise invariant intermittency mechanism displayed in the model.

We next proceeded to a more direct comparison by integrating the model from an initial condition in the DNS and setting the noise perturbation to be $A(0,k) - \alpha_T\nu_T b_t^2 a_k$. Results are shown in Fig. 15. Obviously the model does not do a very good job at tracking the DNS. Note, however, that we are using two different time-stepping schemes, and that both the intermittent model and the flow simulation are sensitive to those. In any case, the model seems able to reproduce the largest time scales of the flow.

Table IV contains low-order statistics of the modes obtained by integrating the model, which show how energy is spread out between the modes. Results were essentially independent of the amount of noise present in the equations. The effects of the quadratic interactions with one exception (mode $a_3$) were correctly captured by the model. The first two modes did give out energy to the other ones, while the last two ones mostly extracted energy. The amplitude of the first modes was slightly overestimated, while that of the highest resolved modes was underestimated. The correlation between any two distinct modes was close to zero, except that between mode $a_1$ and $a_4$. Also note that as a result of the heteroclinic connection, the intensities of $a_1$ and $a_2$ are completely negatively correlated. When the system is close to an equilibrium, the first mode has relatively little energy.

When a burst occurs, this mode becomes unstable and grows while the second mode loses energy. Then the second mode starts growing again, as the first mode shrinks back to zero. We find a significant reminiscence of this in the simulation as the correlation between the most two energetic modes is strongly negative. Overall the repartition of energy in the model is satisfactory, especially as it is in no way imposed by the equations. The first two modes in particular are well (statistically) recovered. However, owing to the crudeness of the truncation, there is a substantial discrepancy between the orderly, recognizable character of ten-dimensional bursts and the complexity displayed in the simulation. To get an idea of how this complexity appears, we decided to examine the effect of including streamwise modes into the model.

B. Adding streamwise modes: The 32-D model

To make the model more realistic, we added one non-zero streamwise mode to the original five-mode truncation and obtained a 16-mode (32-dimensional) model. Such a model has been studied by Sanghi and Aubry. They found windows of intermittency for the parameter $\alpha$ analogous to that present in the 10-D model, i.e., exhibiting heteroclinic cycles. For some values of the parameter $\alpha$, the invariant zero streamwise subspace was attracting, so that streamwise modes decayed to zero. For lower values of $\alpha$, heteroclinic excursions took place in the full streamwise as well as in the zero streamwise subspace.

Our goal is to understand some of the discrepancy between the 10-D model and the physics through examination of what happens to the model in the presence of streamwise modes. The first step is to determine an appropriate value for $\alpha_T\nu_T$, or at least an approximate value, as we are primarily interested in qualitative effects here. Since more scales are included, $\alpha_T\nu_T$ will be lower than in the 10-D model. The amount of energy transfer to the small scales will decrease only slightly; as noted before, transfer to the first streamwise POD modes accounted for only 15% of the total transfer. More importantly, with the inclusion of more modes, the Reynolds stress—hence the mean velocity feedback—will be better estimated, so that less extra dissipation should be needed in the model. We found that for the zero streamwise modes, the cubic feedback terms computed with the 10-D and 32-D model respectively account for 60% and 80% of the total feedback terms. So the mean source term is still somewhat underestimated, as many small scales which contribute to the Reynolds stress are still left out. A crude analysis led us to pick a value around $\alpha = 1.85$. This falls within the intermittency range of the 10-D model, so that if streamwise modes are initially zero, the system will display heteroclinic cycles in the zero streamwise subspace. The question is to determine how these cycles will be affected by the presence of streamwise modes.

Figure 16, which corresponds to a slightly higher $\alpha$ (2.15) than extracted from the simulation, helps clarify the picture. Nonlinear effects remain predominant and the system appears to flip back and forth between two different, yet probably related, types of dynamics, depending on how much the streamwise modes get excited. When the stream-

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**TABLE IV. Statistics for 10-D model.**

<table>
<thead>
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<tbody>
<tr>
<td>$C(Q(0,k),a_k)$</td>
<td>$-0.41$</td>
<td>$-0.60$</td>
<td>$0.80$</td>
<td>$0.99$</td>
<td>$0.99$</td>
</tr>
<tr>
<td>$</td>
<td></td>
<td>a_k</td>
<td></td>
<td>_r$</td>
<td>$1.04$</td>
</tr>
</tbody>
</table>
wise modes remain relatively small in amplitude (from time 0 to about 600 time units in the figure), the zero streamwise subspace displays heteroclinic cycles similar to those in the 10-D model. When the amplitude of the streamwise modes gets large enough, the clearcut pattern of the heteroclinic connection becomes distorted, giving rise to seemingly "chaotic" behavior in the zero streamwise subspace. However, as Fig. 17 shows, the magnitude of the zero streamwise modes evolves in the same cyclic fashion, with the same intermittency period of about 150 outer units.

The realistic case \( \alpha = 1.85 \), shown in Fig. 18, illustrates the second scenario. The intermittency period displayed in Fig. 19 is still about 150 outer units. Figure 18 can be compared to Fig. 20, which shows the evolution of the corresponding modes computed from the numerical simulation of the wall layer at the bottom of the channel—the origin of the time axis was translated in Fig. 20 to make comparison easier. The general features of the time histories appear similar. The streamwise modes move with an average convection velocity of \( 15 \mu_* \) in the model. This is close to the value of \( 12–13 \mu_* \) observed in the simulation, which corresponds to the mean velocity near the top of the layer. As we have seen, in the absence of small scales, the mean profile in the model...
tends to be overestimated, which results in a slightly higher value of the convection velocity.

Table V shows that the energy is relatively well spread out between the modes. The only exception is mode \( a_{10} \), the amplitude of which is much lower in the model than expected. It turns out that the dominant eigenmode for the wavenumber (1,0) makes a negative contribution to the Reynolds stress, i.e., \( \text{Re}[\phi_{10}\phi_{10}^*] \leq 0 \), which yields linear terms \((b^1)\) with negative real part and cubic terms with real positive part. In consequence this mode does not extract energy as well as the other modes. Evidence of negative contribution to the Reynolds stress was also found in Moin and Moser’s data\(^\text{29}\) as well as in Herzog’s experimental data,\(^\text{30}\) albeit at a different wave number. We are not sure about the physical meaning of this, although it is our feeling that modes without any spanwise variations, i.e., unrelated to any rolls or streak patterns, may have little physical relevance for the bursting process. Sanghi and Aubry did not find negative contributions, but they used Herzog’s experimental data at a slightly different wave number. In any case, we found that switching the signs of the contributions of the zero spanwise mode did not alter the global dynamics of the system.

In the 10-D model, the amplitudes of the first two zero streamwise modes were anti-correlated as a consequence of the heteroclinic connection. Table V shows that they are still negatively correlated in the 32-D model, but only partially, with a correlation value \((\sim -0.4)\) which is similar to that found in the simulation. The link between the 10-D model, the 32-D model, and the simulation is made more clear in the comparison of phase portraits shown in Fig. 21. We first examine the relationship between the 10-D model and the “chaotic” regime of the 32-D model [Figs. 21(a) and 21(c)].
Trajectories shown in the figures were obtained with the same value of $\alpha = 1.87$ for both models. Apparently the effect of the streamwise modes is to ‘‘scramble’’ the phase and to a lesser degree the amplitude of the zero streamwise modes. One is reminded of the simple PDF model (see Berkooz et al. [22]) where a random signal was constructed directly from the POD by giving each mode a Gaussian amplitude and a random phase. By analogy, the 32-D model can be seen as a randomization of the 10-D model. However, it is not clear whether and to which extent the heteroclinic cycle still controls the dynamics in the 32-D chaotic regime. Figures 21(a) and 21(c) suggest that there does exist at least some connection. The heteroclinic cycle appears as some limiting curve for the 32-D trajectories, as if the system was still attracted to the cycle, but did not have enough energy (because of the streamwise modes) to do so.

The other point of interest is the relative agreement between trajectories in the DNS and the 32-D model. Even if DNS phase portraits are much less clear-cut than in the 32-D model, both are seen to fill out roughly the same portion of space, itself bounded by the heteroclinic cycle. It would be interesting to examine how the agreement extends to the corresponding velocity fields—the one from the DNS and those reconstructed from the low-dimensional models—over the duration of individual burstlike events. However, such a detailed comparison is out of the scope of the present paper, and will have to be left out for future work. Our conclusion at this stage is that the addition of streamwise modes in the model leads to bursts which relate in some sense to those in the 10-D model but present more real-like features.

VII. CONCLUSION

We have carried out a quantitative evaluation of a POD-based, low-dimensional model for the wall layer ($0 \leq y \leq 55+$), at a relatively low Reynolds number; it is unclear whether the bursting process will still be observed for very high Reynolds numbers. We found that the closure hypotheses underlying the construction of the model were numerically validated. For values of $\alpha$ consistent with the numerical flow simulation, the model was shown to behave intermittently. Furthermore, the intermittency period displayed by the model matched that observed in the simulation. This supports the idea that the low-dimensional model captures essential aspects of the mechanism for the generation of turbulence, although it may not reproduce the detailed physics of the flow; for instance, there was no clear evidence of heteroclinic connections in the simulation. To try and understand this, we simulated a slightly more complex model, and found bursts there that appeared related to the same mechanism, but exhibited less organized features and were in better agreement with the numerical simulation.

Once again we emphasize the fact that the model is based on very simple hypotheses, in particular the influence of unresolved scales is entirely accounted for by additional dissipation. It would be interesting to implement and examine the effect of more sophisticated modeling assumptions, such as those relying on inertial manifolds. In the meantime, we feel that the present study brings a steady confirmation that low-dimensional models are able to capture some key features of near-wall turbulent flows.

ACKNOWLEDGMENT

The author is indebted to John Lumley for many useful suggestions about this manuscript.

APPENDIX: DERIVATION FOR THE MODEL EQUATIONS

Consider the Navier–Stokes equations for the turbulent channel flow:

$$\frac{\partial u_i}{\partial t} = -u_i \frac{\partial u_i}{\partial x_j} - U_i \frac{\partial u_i}{\partial x_j} - u_i \frac{\partial U_i}{\partial x_j} - \frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \left(N(i)\right).$$

Let $k = (l, k)$. We evaluate $\int_0^{\top} \phi_{\top}^{N_l} \phi_{\top}^{N_k} \phi_{\top}^{N_m} \phi_{\top}^{N_n} \frac{\partial u_i}{\partial x_j} \, dy$ and obtain the evolution equation for the mode $a_{\top}^{lkn}$.

- $\frac{\partial u_i}{\partial t}$ becomes $a_{\top}^{lkn}$.

- $-u_i \frac{\partial U_i}{\partial x_j}$ yields linear terms $b_{\top}^{lkn} a_{\top}^{lm}$, where

$$b_{\top}^{lkn} = \int_0^{\top} \frac{\phi_{\top}^{N_l} \phi_{\top}^{N_k} \phi_{\top}^{N_m}}{\nu} \frac{\partial u_i}{\partial x_j} \, dy;$$

- cubic terms $d_{\top}^{lkn} a_{\top}^{pq} a_{\top}^{q} a_{\top}^{l} a_{\top}^{k}$, where

$$d_{\top}^{lkn} = \frac{1}{\nu} \int_0^{\top} \frac{\phi_{\top}^{N_l} \phi_{\top}^{N_k} \phi_{\top}^{N_m}}{\nu} \phi_{\top}^{N_q} \phi_{\top}^{N_k} a_{\top}^{q} \, dy.$$

- $-U_i \frac{\partial u_i}{\partial x_j}$ yields linear terms $b_{\top}^{lkn} a_{\top}^{lm}$, where

$$b_{\top}^{lkn} = \int_0^{\top} \frac{\phi_{\top}^{N_l} \phi_{\top}^{N_k} \phi_{\top}^{N_m}}{\nu} \frac{\partial u_i}{\partial x_j} \, dy;$$

- cubic terms $d_{\top}^{lkn} a_{\top}^{pq} a_{\top}^{q} a_{\top}^{l} a_{\top}^{k}$, where

$$d_{\top}^{lkn} = \frac{1}{\nu} \int_0^{\top} \frac{\phi_{\top}^{N_l} \phi_{\top}^{N_k} \phi_{\top}^{N_m}}{\nu} \phi_{\top}^{N_q} \phi_{\top}^{N_k} a_{\top}^{q} \, dy.$$

Define

$$b_{\top} = b_{\top} \phi_{\top}^{N_k} + b_{\top} \phi_{\top}^{N_k};$$

$$d_{\top} = d_{\top} \phi_{\top}^{N_k} + d_{\top} \phi_{\top}^{N_k}.$$

In the case where $l = 0, b_{\top} = d_{\top} = 0$.

Denote

$$\Omega = \begin{cases} \sqrt{-1} k_j & \text{if } j = 1 \text{ or } j = 3, \\ d/dy & \text{if } j = 2. \end{cases}$$

- $\nu \frac{\partial u_j}{\partial x_k \partial x_k}$ yields linear terms $b_{\top}^{lkn} a_{\top}^{lm}$, where

$$b_{\top}^{lkn} = \int_0^{\top} \nu \Omega^2 \phi_{\top}^{N_l} \phi_{\top}^{N_k} \phi_{\top}^{N_m} \phi_{\top}^{N_n} \, dy.$$
\[-u' \partial u'/\partial x_j\] is decomposed into a resolved and an unresolved part.

- The resolved part gives quadratic terms \( \epsilon_{k'k-k}^{pq} \), where

\[
\epsilon_{k'k-k}^{pq} = \int_0^{\text{top}} \partial \Omega_k^{ij} \partial j_k^{\alpha ij} \Omega_k^{\alpha q} \, dy.
\]

- The unresolved part can be decomposed into an anisotropic and isotropic part (pseudo-pressure term).

*The anisotropic component of the unresolved part is modeled by a linear term \(-\alpha \Phi_k \Phi_k^q\).

*The isotropic component of the unresolved part is modeled using a Heisenberg-like approximation and yields extra quadratic interactions (see Ref. 25 for details). The contribution of this pseudo-pressure term was found to be negligible.

\[-(1/\rho) (\partial p/\partial x_i)\] yields a term evaluated at the edges of the domain \(-[\Phi_k \Phi_k^{2n}]_{y=0}^{\text{top}}\). This term represents the influence of the outer layer and behaves like a stochastic forcing term. Its magnitude was found to be very small and it was omitted from the present analysis.