MODELING OF A 1D GAS FLOW AT LOW MACH REGIME IN A THERMOSIPHON

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Introduction

Context
Transport and distribution of liquified natural gas through pipeline networks

Main goals
- Construct a 1D low Mach model
- Validation through a thermosyphon
Previous works


» Mach number equal to zero

» 1D averaged equations for pipes

» Take in consideration the gravity term

» Isothermal assumption

» Construction of a low Mach model.

» Different context
PART I

PHYSICAL & MATHEMATICAL MODELING
1D Averaged equations

1D COMPRESSIBLE NAVIER-STOKES EQUATIONS

For ideal gases

\[
\begin{align*}
\frac{\partial S\rho}{\partial t} + \frac{\partial S\rho u}{\partial x} &= 0 \\
\frac{\partial S\rho u}{\partial t} + \frac{\partial S\rho u^2}{\partial x} + \frac{\partial SP}{\partial x} &= -\tau\pi D - \rho S g \sin \theta \\
\rho C_p \left( \frac{\partial ST}{\partial t} + u \frac{\partial ST}{\partial x} \right) &= D_t(SP) + u\tau \pi D - 2\pi R q_w \\
\rho &= \frac{P}{rT}
\end{align*}
\]
The dimensionless equations

\[ Ma^2 = \frac{\rho_c u_c^2}{\gamma P_c} \]

\[ \frac{\partial \tilde{S} \tilde{ \rho}u}{\partial \tilde{\tau}} + \frac{\partial \tilde{S} \tilde{ \rho}u^2}{\partial \tilde{x}} + \frac{1}{\gamma Ma^2} \frac{\partial \tilde{S} \tilde{P}}{\partial \tilde{x}} = -\frac{f}{2} \pi \tilde{\rho} u^2 \tilde{D} - \frac{1}{Fr^2} \tilde{S} \tilde{\rho} g \sin \theta \]

\[ Re = \frac{u_c L_c \rho_c}{\mu_c} \]

\[ Fr^2 = \frac{u_c^2}{L_c g} \]

\[ Pr = \frac{\mu_c C_{pc}}{k_c} \]
The low Mach model

\[ Ma \rightarrow 0 \]
\[ (Ma \approx 10^{-2}) \]

Series expansion around \( Ma \)

\[ \tilde{u}(x, t) = \tilde{u}_0(x, t) + Ma\tilde{u}_1(x, t) + O(Ma^2) \]
\[ \tilde{T}(x, t) = \tilde{T}_0(x, t) + Ma\tilde{T}_1(x, t) + O(Ma^2) \]
\[ \tilde{P}(x, t) = \tilde{P}_0(x, t) + Ma\tilde{P}_1(x, t) + Ma^2\tilde{P}_2(x, t) + O(Ma^3) \]

**Order 0**

\[ \frac{\partial \tilde{S}\tilde{\rho}_0}{\partial \tilde{t}} + \frac{\partial \tilde{S}\tilde{\rho}_0\tilde{u}_0}{\partial \tilde{x}} = 0 \]

**Dynamic pressure** \( \Pi \)

\[ \frac{\partial \tilde{S}\tilde{\rho}_0\tilde{u}_0}{\partial \tilde{t}} + \frac{\partial \tilde{S}\tilde{\rho}_0\tilde{u}_0^2}{\partial \tilde{x}} + \frac{1}{\gamma} \frac{\partial \tilde{S}\tilde{P}_2}{\partial \tilde{x}} = -\frac{f}{2} \pi\tilde{\rho}_0\tilde{u}_0^2\tilde{D} - \frac{1}{Fr} \tilde{\rho}_0 S g \sin\theta \]

**Order -1**

\[ \partial_x \tilde{P}_1 = 0 \]
\[ \partial_x \tilde{P}_0 = 0 \]

**Order -2**

**Series expansion**

\[ \tilde{\rho}_0 \tilde{C}_p \left( \frac{\partial \tilde{S}\tilde{T}_0}{\partial \tilde{t}} + \tilde{u}_0 \frac{\partial \tilde{S}\tilde{T}_0}{\partial \tilde{x}} \right) = \frac{\gamma - 1}{\gamma} \tilde{S}\tilde{P}_0' - \frac{2\pi \tilde{R}}{PrRe} \tilde{q}_w \]

**Thermodynamic pressure** \( P \)

\[ Ma \approx 10^{-2} \]

\[ Ma \rightarrow 0 \]
The low Mach model

BEFORE:
3 variables
1 pressure
3 equations

NOW:
4 variables
2 pressures
3 equations

\[
\frac{\partial S u}{\partial x} = - \frac{S}{\gamma P} P'(t) - \frac{2\pi R (\gamma - 1)}{\gamma P} q_w
\]

\[
\partial_t u + u \partial_x u + \frac{\partial_x \Pi}{\rho} = - \frac{f}{2} \pi u \frac{D}{S} - g \sin \theta
\]

\[
\rho C_p \left( \frac{\partial S T}{\partial t} + u \frac{\partial S T}{\partial x} \right) = S P'(t) - 2\pi R q_w
\]

\[
q_w := h(T - T_{ref})
\]
Equations for $P, \Pi$

**Equation for $P(t)$**

$$P'(t) = -\frac{2\pi R (\gamma - 1)}{S|\Omega|} \int_\Omega q_w$$

**Equations for $\Pi(x, t)$**

$$\partial_t u + u \partial_x u + \frac{\partial_x \Pi}{\rho} = -\frac{f}{2\pi u^2} \frac{D}{S} - g \sin \theta$$

$$\frac{\partial u}{\partial x} = -\frac{1}{\gamma P} P'(t) - \frac{2\pi R (\gamma - 1)}{S\gamma P} q_w := \eta$$

$$\partial_x \left( \frac{\partial_x \Pi}{\rho} \right) = \partial_t \eta + u \partial_x \eta + \eta^2 + f\pi u^2 \frac{D}{S} - g\delta(x - nL)$$
Comparison with Boussinesq

**BOUSSINESQ**

\[ \frac{\partial Su}{\partial x} = 0 \]

\[ \frac{\partial SP_2}{\partial x} = -\frac{f}{2} \pi \rho_{ref} u^2 D + \alpha \rho_{ref} (T - T_{ref}) S g \sin \theta \]

\[ \rho C_p u \frac{\partial ST}{\partial x} = -2\pi R q_w \]

\[ P_0 = \text{cst} \]

**LOW MACH MODEL**

\[ \frac{\partial Su}{\partial x} = -\frac{S}{\gamma P_0} P_0'(t) - \frac{2\pi R (\gamma - 1)}{\gamma P_0} q_w \]

\[ \frac{\partial SP_2}{\partial x} = -\frac{f}{2} \pi \rho u^2 D - \rho g S \sin \theta \]

\[ \rho C_p u \frac{\partial ST}{\partial x} = SP_0'(t) - 2\pi R q_w \]

\[ P_0'(t) = -\frac{2\pi R (\gamma - 1)}{S |\Omega|} \int_{\Omega} q_w \]
The Thermosyphon

REFERENCE ANALYTICAL SOLUTIONS:

\[ T(x) = T_{\text{ref}} + (T_0 - T_{\text{ref}})e^{-\frac{2\pi R(\gamma - 1)h}{\gamma P \Gamma} x} \]

\[ Q(x) = \Gamma T(x) \]

\[ \Pi(x) = \frac{b P \Gamma}{r B h T_{\text{ref}}} \left( \ln T(x) - \ln T_0 \right) + \frac{P}{r T_{\text{ref}}} \left( a T_{\text{ref}} \Gamma - b \right) x + \Pi_0 \]

\[ \Gamma := \frac{Q_0}{T_0} = \frac{Q_1}{T_1} \]

\( \Gamma \) found through dichotomy & iterative methods
PART II

NUMERICAL MODELING
Numerical scheme for closed domains

Algorithm for $P$

$$p^{n+1} = p^n - \Delta t \frac{2\pi R (\gamma - 1)}{S |\Omega|} \int_{\Omega} q_{w}^{n+1}$$

Characteristics for $T$

$$T_i^{n+1} = \frac{\hat{T}_i^n + \Delta t \frac{2\pi R (\gamma - 1) T_i^n}{\gamma P^n S} \left( -\frac{\gamma - 1}{|\Omega|} \int_{\Omega} q_{w}^n + hT_{ref} \right)}{1 + \Delta t \frac{2\pi R (\gamma - 1) hT_i^n}{\gamma P^n S}}$$

Splitting algorithm

Prediction and projection steps to compute $u$ and $\Pi$, coupled with a linear system solving for $\Pi$
The splitting procedure

\[ \partial_t u = \frac{u^{n+1} - u^*}{\Delta t} + \frac{u^* - u^n}{\Delta t} \]

\[ \partial_x u^{n+1} = \eta^{n+1} \rightarrow u \partial_x u \approx u^n \eta^{n+1} \]

\[ \frac{u^* - u^n}{\Delta t} + u^n \eta^{n+1} = -f \pi u^* \frac{R}{S} \rightarrow u^* = u^n \frac{1 - \Delta t \eta^{n+1}}{1 + \Delta t f \pi \frac{R}{S}} \]

\[ \frac{u^{n+1} - u^*}{\Delta t} = -\frac{1}{\rho} \frac{\partial \Pi}{\partial x} - g \sin \theta \rightarrow u^{n+1} = u^* + \Delta t \left( -\frac{1}{\rho} \frac{\partial \Pi}{\partial x} - g \sin \theta \right) \]

\[ \partial_x \left( \frac{1}{\rho} \frac{\partial \Pi}{\partial x} \right) = -g \delta(x - nL) + \frac{1}{\Delta t} \left( \partial_x u^* - \eta^{n+1} \right) \]
Dirac Deltas: problem setting

\[
\begin{cases}
-\partial_x (k(x)\partial_x \Pi) = f(x) + \sum_{i=1}^{D} \alpha_i \delta_{\hat{x}_i} \\
\Pi(0) = 0 \\
\Pi(L) = 0
\end{cases}
\]

We need \(\alpha_i\)

\[
\sin \theta(x) := \begin{cases}
1 & \text{if } x < l \land x > 4l \\
-1 & \text{if } 2l < x < 3l \\
0 & \text{elsewhere}
\end{cases}
\]

\(\hat{x}_1 = l, \quad \hat{x}_2 = 2l, \quad \hat{x}_3 = 3l, \quad \hat{x}_4 = 4l\)

\(\alpha_1 = -1, \quad \alpha_2 = -1, \quad \alpha_3 = 1, \quad \alpha_4 = 1\)
Dirac Deltas: scheme I

Definition of the Flux

Scheme

\[ F_{i+\frac{1}{2}} = -\frac{\Pi_{i+1} - \Pi_i}{\Delta x} \approx -\partial_x \Pi(x_{i+\frac{1}{2}}) \]

\[ F_{i-\frac{1}{2}} = -\frac{\Pi_i - \Pi_{i-1}}{\Delta x} \approx \partial_x \Pi(x_{i-\frac{1}{2}}) \]

Source term average

\[ \hat{f}_i := \frac{1}{\Delta x} \int_{K_i} f(x)dx \]

\[ F_{\frac{1}{2}} := -\frac{\Pi_1}{\Delta x}, \quad F_{N+\frac{1}{2}} := \frac{\Pi_N}{\Delta x} \]
Dirac Deltas: scheme II

New definition of flux

\[ F^+_{j+\frac{1}{2}} = \frac{k_{j+1}}{k_{j+1} + k_j} \alpha + F^+_{j+\frac{1}{2}} \]
\[ F^-_{j+\frac{1}{2}} = -\frac{k_j}{k_{j+1} + k_j} \alpha + F^-_{j+\frac{1}{2}} \]

New equation

\[ [-k(x)\partial_x \Pi](x^+) - [-k(x)\partial_x \Pi](x^-) = \alpha \]

\[ F^+_{j+\frac{1}{2}} - F^-_{j+\frac{1}{2}} = \alpha \neq 0 \]

Scheme

\[ \begin{cases} F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} = \Delta x \hat{f}_i & \forall i \neq j \\ F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = \Delta x \hat{f}_j + \frac{k_j}{k_{j+1} + k_j} \alpha \\ F_{j+\frac{3}{2}} - F_{j+\frac{1}{2}} = \Delta x \hat{f}_{j+1} + \frac{k_{j+1}}{k_{j+1} + k_j} \alpha \end{cases} \]

Discontinuous flux!
Dirac Deltas: solution matrix

\[
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
D_{\frac{1}{2}} & D_{\frac{1}{2}} & D_{\frac{1}{2}} & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & D_{N-\frac{3}{2}} & D_{N-\frac{3}{2}} & D_{N-\frac{3}{2}} \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
\]

\[
D_{i+\frac{1}{2}} = 2k_i \left( \frac{k_{i-1}}{k_{i-1} + k_i} + \frac{k_{i+1}}{k_{i+1} + k_i} \right)
\]

\[
D_{i+\frac{1}{2}}^{-1} = -2k_i \frac{k_{i-1}}{k_{i-1} + k_i}
\]

\[
D_{i+\frac{1}{2}}^1 = -2k_i \frac{k_{i+1}}{k_{i+1} + k_i}
\]

\[A\] is a tridiagonal matrix $\rightarrow$ we use Thomas algorithm $\rightarrow \mathcal{O}(n)$
PART III

RESULTS
Errors

![Graphs showing errors and computing time vs. Δx]
Profiles of $T$, $u$, $\Pi$, $P$
Dimensionless numbers

We therefore have a Laminar Flow
Current research status
PART VI

CONCLUSIONS
Achievements

- Our **MODEL** is an extension of the Boussinesq model:
  - capable of describing $\Delta P$
  - Valid at large $\Delta T$
  - Tends to Boussinesq as $\Delta T \to 0$

- Our **ALGORITHM** is capable of:
  - Dealing with abrupt changes in sinth
  - Describing the transmission conditions at the junction
  - Providing numerical simulation of the first order in time and space

- We constructed a **REFERENCE ANALYTICAL SOLUTION** for the thermosyphon
Future developments

» Take into consideration an inlet for closed domains
» Consider different rheology

In progress:

✓ Compare numerical solutions to experimental results
✓ Study more complex junctions
✓ Examine mixtures of gases in different concentrations
✓ Enrich the code by allowing different physical/geometrical settings
Contributions:

» Developed a 1D low Mach model for pipeline networks extending the Boussinesq model
» Found laminar steady analytical solutions to the thermosyphon
» Conceived a numerical algorithm for treating the Dirac deltas
» Constructed a numerical algorithm for the simulation of gas flow in pipeline networks of the first order in time and space
» Conceived an industrial code easily applicable to engineering settings for the simulation of temperature-driven flows

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