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| **Complete List of Authors:** | Herreman, Wietze; LIMSI, CNRS, Université Paris-Sud, Université Paris-Saclay, Mechanics  
Nore, Caroline; LIMSI, CNRS UPR 3251,  
Guermond, Jean-Luc; Texas A&M University  
Cappanera, Loïc; Rice University, Department of Computational and Applied Mathematics  
Weber, Norbert; Helmholtz-Zentrum Dresden - Rossendorf  
Horstmann, Gerrit Maik; Helmholtz-Zentrum Dresden - Rossendorf e.V., Institute of Fluid Dynamics |
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Metal pad roll instability in cylindrical reduction cells

W. Herreman\textsuperscript{1,}\textsuperscript{†}, C. Nore\textsuperscript{1}, J.-L. Guermond\textsuperscript{2}, L. Cappanera\textsuperscript{3}, N. Weber\textsuperscript{4} and G. M. Horstmann\textsuperscript{4}

\textsuperscript{1}LIMSI, CNRS, Université Paris-Sud, Université Paris-Saclay, Orsay, F-91405, France
\textsuperscript{2}TAMU, Texas A\&M, College station, USA
\textsuperscript{3}CAAM, Rice University, 6100 Main St., Houston, TX 77005-1827, USA
\textsuperscript{4}Helmholtz-Zentrum Dresden - Rossendorf, Bautzner Landstr. 400, 01328 Dresden, Germany

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We formulate a precise theoretical model for the metal pad roll instability of gravity waves in a cylindrical reduction cell. This model is new and goes beyond the simplifying assumptions that are usually made in metal pad roll theories: it does not require the layers to be shallow, it includes viscous and magnetic damping, it includes surface tension, it is valid for any combination of metal, and it captures both horizontal and vertical magnetic field effects. We confront the calculated viscous damping rates to experimental measurements and the linear stability theory to direct numerical simulations done with two independent numerical solvers, SFEMaNS and OpenFOAM, and in both cases find excellent agreements.

1. Introduction

The metal pad roll instability is a magnetohydrodynamical phenomenon that disturbs the production of aluminium in Hall-Heroult reduction cells. A small background magnetic field that reigns in the aluminium factory can interact with the electrolysis current to create a magnetic coupling between transverse gravity waves. Above a minimal electrolysis current or beneath a minimal cryolite layer depth, this coupling can drive rotating gravity waves on the interface between the cryolite and the aluminium. This is undesirable as such waves can cause short-circuits between the carbon anodes and the aluminium bath which makes the industrial process less efficient.

The first physical descriptions of the metal pad roll instability go back to Urata et al. (1976) and Sele (1977), but many other models have been developed in order to better understand the metal pad roll instability in aluminium reduction cells (Sneyd 1985; Moreau & Ziegler 1986; Davidson & Boivin 1992; Ziegler 1993; Sneyd & Wang 1994; Bojarevics & Romero 1994; Davidson & Lindsay 1998; Lukyanov et al. 2001; Molokov et al. 2011). With their very shallow character (several meters in lateral extend, only a few centimeters in height), Hall-Heroult cells can be accurately modelled using shallow layer expansions (see Bojarevics & Romero 1994). Most often these cells are so large that all the capillary, viscous and magnetic dissipation effects are ignored at first order, and inviscid instability criteria then involve only one non-dimensional parameter and are all of the form $\beta > \beta_c$. Here $\beta$ is a non-dimensional number defined below (see (2.56)) and $\beta_c$ only depends on the geometry of the cell and on the vertical magnetic field and electrolysis current.

Although the linear theory can explain the physical origin of the instability, it is not

\textsuperscript{†} Email address for correspondence: wietze@limsi.fr
adapted to predict the nonlinear development or the amplitudes of the destabilized waves. Several two-dimensional and nonlinear shallow models have therefore been derived and perfected over the years to account for the complex magnetic fields that may occur in industrial cell configurations (Bojarevics & Romero 1994; Zikanov et al. 2004; Bojarevics & Pericleous 2006; 2008). These models are so economical to run that they can effectively be used to design stabler reductions cells. As numerical resources increased over time, it also became possible to do direct numerical simulations of the full three-dimensional nonlinear dynamics of reduction cells. This approach is much more computationally demanding and was initiated by Gerbeau et al. (2003; 2004; 2006) who used a finite element ALE-method. Not much later, commercial codes ANSYS & CFX were interfaced in Severo et al. (2005; 2008). A finite volume level-set method was developed by Munger & Vincent (2006a; c) and a finite-element method with mobile grid was developed in Flueck et al. (2009); Steiner (2009); Flueck et al. (2010). Each of these advanced numerical tools have been carefully tested and applied to aluminium reduction cells, but true quantitative comparisons between direct numerical simulations and theoretical models remain relatively scarce for the following reasons: (i) theoretical models often ignore dissipative effects, which cannot be ignored in DNS; (ii) most theories are dedicated to shallow cells but not all the simulations are done in shallow cells; (iii) many direct numerical simulations focus on realistic cell geometries for which no simple theoretical model is available. As a result most metal pad roll theories are rather disconnected from what happens in direct numerical simulations. It is our opinion that there are too few clear overlaps between theory and direct numerical simulations, making it difficult to estimate what theoretical mechanisms are operational in which simulations.

Our first motivation for this study finds here its origin. In this article, we provide a detailed description of the metal pad roll phenomenon combining a precise theory with well-resolved direct numerical simulations and even some (hydrodynamical) experiments. This ambitious programme inevitably results in a long article.

Our second motivation for this study is to be found in the context of liquid metal batteries. Lately, the metal pad roll phenomenon has been studied in these three-layer systems (Zikanov 2015; Weber et al. 2017a; b; Bojarevics & Tucs 2017; Horstmann et al. 2018a; Molokov 2018; Zikanov 2018). Although we can imagine that future large scale liquid battery systems may eventually become as shallow as Hall-Heroult reduction cells, present day prototypes are at laboratory scale and are certainly not always shallow. To interpret the numerical simulations of Weber et al. (2017a; b) and Horstmann et al. (2018a) a new theory that is non-shallow, viscous and includes magnetic damping effects and capillarity seemed necessary. But first we needed to learn how to manage only two fluid layers, which is the topic of the present paper.

Our third motivation is more a numerical one. All the simulations presented here are done with two different multiphase MHD solvers, SFEMaNS and OpenFOAM that arrive as relatively new players in the area of metal pad roll simulations. Our first solver SFEMaNS is being developed since 2001 and was initially designed to solve MHD problems in axisymmetric geometries. The discretization is done using Fourier expansions in the azimuthal direction and finite elements in the \((r, z)\)-plane. First designed to study dynamo problems in axisymmetric domains (see Guermond et al. (2007; 2009)), the code has been modified to account for non-axisymmetric geometries (Nore et al. 2016). More recently, it has been further upgraded to solve multiphase MHD problems using a level-set method that is fully explained in Cappanera et al. (2018). In Herreman et al. (2015) we used SFEMaNS to study the Tayler instability in liquid metal battery systems and some preliminary simulations on metal pad roll instability are mentioned in Cappanera et al. (2018). Our second solver is based on the finite volume library OpenFOAM (Weller
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et al. 1998). Since 2010 this code is capable of solving quasi-static MHD and it was used to study Tayler instability (Weber et al. 2013, 2014) and electrovortex flow (Weber et al. 2018; Ashour et al. 2018). A volume-of-fluid model is used to implement multiple liquid phases and has allowed to perform several studies on metal pad roll instability in liquid metal batteries (Weber et al. 2017a,b; Horstmann et al. 2018a). Although both codes have been used in previous studies, we still have not passed a fine benchmark involving both codes at the same time. We found that the metal pad roll instability is perfect for this type of test and evidently needed a precise theoretical model to compare with.

In § 2, we formulate the theoretical model for the metal pad roll instability in a cylindrical reduction cell. Cylindrical cells are perhaps not realistic for industrial applications but this configuration has been well investigated in the past both from the theoretical point of view (Davidson & Lindsay 1998; Lukyanov et al. 2001) and the numerical point of view (Gerbeau et al. 2003; Steiner 2009). The proposed model is new and overcomes simplifying assumptions that are usually made in metal pad roll theories: it is non-shallow, includes magnetic damping, includes surface tension, is valid for any combination of conducting fluids and includes viscous damping. The price to pay for this generality is that the theory is a perturbation analysis with limited range of validity. We will demonstrate that even large cells can be unstable in the range covered by our theory, but the true novelties will be visible in small cells, which are much more prone to viscous and magnetic damping and capillary effects.

In § 3, we discuss some purely hydrodynamical experiments that serve to test the theoretical viscous damping rates estimated in § 2. These datapoints are by-products of a novel experimental study on orbital sloshing in two and three layer fluid systems from Horstmann et al. (2018b). Orbital sloshing corresponds to prescribing a circular translation to a container (used to swirl for example a glass of wine) and it drives exactly the same wave mode that is commonly associated with the metal pad roll instability (see for example Reclari et al. 2014; Bouvard et al. 2017; Moisy et al. 2018). In this experimental study, we form a stable interface fulfilling in good approximation the idealised free-moving contact line boundary condition assumed in our theory by using Paraffinum Perliquidum and Wacker® silicone oil AK 35. In this section we compare the theoretical predictions with measured interfacial damping rates for different layer heights and find very good agreements.

In § 4, we apply the theoretical model to two different reduction cells, a small laboratory-scale cylindrical reduction cell and a large industrial-scale cell. For the large cells, we limit ourselves to applying the theoretical model, demonstrating here that perturbation methods can be used even in such large cells. The use of the small cell is more academic and is briefly discussed in Flueck et al. (2009) and Steiner (2009) as a test-case for their multiphase MHD solver. With a radius of only $R = 0.035$ m, the small lab-scale cell can only become unstable when very large ambient magnetic fields are present, far above what can be achieved in aluminium factories, but close to what we had in the experiments of Pedchenko et al. (2009, 2016). The biggest advantage and interest of the small cell geometry is that very well resolved direct numerical simulations can be done, thereby allowing us to compare gravity wave frequencies, viscous dissipation rates, magnetic dissipation rates and finally metal pad roll instability growth rates for various parameter sets and waves and using different numerical resolutions. At the end of this section, we also apply our model to a $\text{Mg}||\text{NaCl} – \text{KCl} – \text{MgCl}_2||\text{Sb}$ liquid metal battery. This is a three layer system that is not modelled by our theory, but since the bottom fluid layer is very heavy, it almost does not participate to the destabilization/stabilization mechanism of the cell. With some minor technical transformations we can apply our
Figure 1. Sketch of cylindrical model cell and notations. (a) Two fluid layers with different densities $\rho_1, \rho_2$ and respective heights $H_1, H_2$ are stably superposed in a cylindrical container of radius $R$ due to gravity $g$. The layers have different electrical conductivity and kinematic viscosity denoted as $\sigma_i, \nu_i, i = 1, 2$. A vertical electrical current $J e_z$ runs through the cell (red full line) which generates an azimuthal magnetic field $B_\theta$ (red dashed line). We also suppose a homogenous and vertical background magnetic field $B_z$ (bleu dashed line). (b) We use cylindrical coordinates $(r, \theta, z)$ and denote $V_i, i = 1, 2$ for both fluid regions. $\Sigma_i, i = 1, 2$ refer to the rigid boundaries, $S$ to the free interface at rest, $M$ to the meniscus region. $n_i$ is the external normal on both fluid regions.

theory to model the metal pad roll instability in this battery and compare it to previously obtained numerical results (Weber et al. 2017a).

In § 5, we conclude and discuss some perspectives of our study.

2. Theoretical analysis

2.1. Cylindrical reduction cell at equilibrium

The cylindrical cell is sketched in figure[1] We align the axis of symmetry with the $z$-axis of a cylindrical coordinate system $(r, \theta, z)$ with unit vectors $(e_r, e_\theta, e_z)$. At equilibrium, the cryolite and aluminium layers (subscripts $i = 1, 2$) occupy the volumes

$$
V_1 : (r, \theta, z) \in [0, R] \times [0, 2\pi] \times [0, H_1]
$$

$$
V_2 : (r, \theta, z) \in [0, R] \times [0, 2\pi] \times [-H_2, 0],
$$

where $R$ is the radius of the cell and $H_i$ is the height of both layers. We denote $\delta V_i$ the boundary of each fluid domain and split each boundary $\delta V_i$ as follows $\delta V_i = \Sigma_i \cup S$, with $\Sigma_i$ the rigid boundary and $S$ the interface at rest. Here $n_i$ is the outward normal on $\delta V_i$.

The electrical conductivity, density, kinematic viscosity of both fluids are denoted by $\sigma_i, \rho_i, \nu_i$, respectively. The surface tension of the interface between the two fluids is denoted by $\gamma_{12}$. We suppose that the interface touches the lateral wall with a contact angle of $90^\circ$ and that it slides freely. This is a strong hypothesis that is necessary to allow for an equilibrium state with a flat interface. Capillarity is rarely considered in metal pad roll theories since the capillary length is always very small compared to the system size in realistic cells. But capillarity may be relevant in small cells, and since it can be easily included in the model, we decide to consider its effect.

If the cell is at (magnetohydrodynamical) equilibrium, both fluids are at rest. The fluids
are subject to gravity $g = -g \mathbf{e}_z$. A homogenous electrical current density $\mathbf{J} = J \mathbf{e}_z$ runs through the cell (in reduction cells, $J < 0$ always). A magnetic field $\mathbf{B} = B_0 \mathbf{e}_\theta + B_z \mathbf{e}_z$ is present. The azimuthal (or horizontal) field $B_0 = \mu_0 J/2$ increases radially and is self-generated by the electrical current in the cell. The vertical component of the magnetic field $B_z$ is supposed to be constant and entirely due to an external source. The above state is a magnetohydrodynamical equilibrium configuration if the hydrodynamic pressure is $P_i = P_* - \rho_i g z - \mu_0 J^2 r^2/4$ and the electrical potential is $\Phi_i = \Phi_* - Jz/\sigma_i$ with $P_*, \Phi_*$ arbitrary and constant offsets.

2.2. Linearised problem in the quasi-static limit

From many previous studies we know that the equilibrium state can become unstable. We perform a linear stability analysis that takes into account viscous dissipation, magnetic dissipation and surface tension, but we do not make the shallow layer approximation. We start by linearizing the governing equations about the equilibrium state. Denoting $\mathbf{B} = \max(\mu_0 J R, B_z)$ the typical scale for the magnetic field strength, we will suppose that the Lundquist numbers $Lu_i$ in both layers remain very low:

$$Lu_i = \sigma_i \mu_0 BR/\sqrt{\rho_i \mu_0} \ll 1. \quad (2.2)$$

This hypothesis allows us to use the quasi-static approximation of MHD, in which the electrical field derives from the electrical potential. Denoting by $(u_i, p_i, b_i, j_i, \varphi_i)$ perturbations for the velocity, the pressure, the magnetic induction, the current density and the electrical potential, we find the linearized perturbation equations of the magnetohydrodynamical stability problem to be

$$\rho_i \partial_t u_i + \nabla p_i = \mathbf{J} \times \mathbf{b}_i + j_i \times \mathbf{B} + \rho_i \nu_i \nabla^2 u_i, \quad (2.3a)$$

$$\nabla \cdot u_i = 0, \quad (2.3b)$$

$$j_i = \sigma_i (-\nabla \varphi_i + u_i \times \mathbf{B}), \quad \nabla \cdot j_i = 0, \quad (2.3c)$$

$$\nabla \times b_i = \mu_0 j_i, \quad \nabla \cdot b_i = 0, \quad (2.3d)$$

for $i = 1, 2$. The essential inviscid hydrodynamical boundary condition on the rigid boundaries $\Sigma_i$ is that of impermeability

$$u_i \cdot n_i = 0|_{\Sigma_i}, \quad (2.4)$$

to which we must add $u_i \times n_i = 0|_{\Sigma_i}$ when the viscous effects are accounted for. On the interface $S$, which we locate at $z = \eta(r, \theta, t)$, the essential inviscid boundary conditions are

$$\partial_t \eta = u_{1,z}|_{z=0}, \quad (2.5a)$$

$$\partial_t \eta = u_{2,z}|_{z=0}, \quad (2.5b)$$

$$-\gamma_1 |d \nabla^2 \eta + (\rho_2 - \rho_1) g \eta = p_2|_{z=0} - p_1|_{z=0}. \quad (2.5c)$$

When the viscous effects are accounted for we must add the continuity of the tangential components of the velocity and the continuity of the tangential viscous stresses (explicit formulas are not required in the article). For the electrical boundary conditions on the rigid boundaries $\Sigma_i$, we will suppose

$$j_{1,r}|_{r=R} = 0, \quad (2.6a)$$

$$j_{2,r}|_{r=R} = 0, \quad (2.6b)$$

$$\varphi_1|_{z=H_1} = 0, \quad (2.6c)$$

$$j_2,z|_{z=-H_2} = 0. \quad (2.6d)$$
The cylindrical wall is always insulating. The top boundary is treated as an electrical contact with a perfect conductor (iso-potential surface), mimicking the fact that the Carbon anode is a very good conductor compared to cryolite. The bottom boundary is treated as an electrical contact with an insulating material, mimicking the fact that the steel cathode is weakly conducting compared to the liquid aluminium. These idealised boundary conditions have been used by many authors and are relevant to reduction cells. On the free interface $S$, we need to satisfy

$$0 = j_{1,z}|_{z=0} - j_{2,z}|_{z=0},$$

$$J(\sigma_2^{-1} - \sigma_1^{-1})\eta = \varphi_2|_{z=0} - \varphi_1|_{z=0}. \tag{2.7a}$$

Equation (2.7b) plays an essential role for the metal pad roll instability, since it expresses how electrical potential perturbations $\varphi_i$ (and so currents $j_i$) relate to the motion of the interface. Finally, for the magnetic induction driven by the electrical currents, we will suppose the following simplifying boundary condition on the cylindrical boundary:

$$e_r \times b_i|_{r=R} = 0. \tag{2.8}$$

Boundary conditions at other places are not required in the analysis. This condition simplifies the analysis as it will allow us to avoid the explicit calculation of the magnetic field, which is never an easy task in finite fluid domains surrounded by a current-free exterior. A similar boundary condition was used in a previous study on the Tayler instability [Herreman et al. 2015] and was shown therein to have little impact on the threshold and the nature of this instability.

The linearized problem is now completely defined and so we can look for linear eigenmodes. Due to the stationarity of the equilibrium state, we look for fundamental solutions with a simple exponential time-dependence. More precisely, if $f$ represents any of the relevant fields or components, we propose the ansatz

$$f = \hat{f} e^{st}, \tag{2.9}$$

and try to find the admissible (complex) eigenmodes $\hat{f}$ and (complex) eigenvalues $s \in \mathbb{C}$. We call Re$(s)$ the growth rate and Im$(s)$ the frequency. If Re$(s) > 0$, the eigenmode is linearly unstable and if Im$(s) \neq 0$ the solution is wave-like.

Using a numerical approach it is possible to solve this linear eigenvalue problem exactly. In the following parts of this theoretical section, we use a perturbative approach. We model how free inviscid gravity waves are weakly stabilized, destabilized or detuned in frequency, by weak magnetohydrodynamical and viscous effects.

### 2.3. Free inviscid gravito-capillary waves

Our perturbative approach is based on the free inviscid gravito-capillary waves that form a family of eigensolutions in absence of electrolysis current ($J = 0$), vertical magnetic field ($B_z = 0$), and without viscosity ($\nu = 0$). In that case, the following hydrodynamical field

$$[u_i, p_i, \eta] = [\hat{u}_i, \hat{p}_i, \hat{\eta}] e^{i\omega t}, \tag{2.10}$$

is a solution to the linearized problem defined above. The hydrodynamical fields $\hat{u}_i, \hat{p}_i, \hat{\eta}$ correspond to the classical potential wave solution (see Lamb 1945):

$$\begin{bmatrix} \hat{u}_i \\ \hat{p}_i \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} \nabla \hat{\phi}_i \\ -\rho_i(i\omega)\hat{\phi}_i \\ \partial_z \hat{\phi}_i/(i\omega) \end{bmatrix}, \tag{2.11}$$
with

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\eta} \end{bmatrix} = A \begin{bmatrix} \cosh(k(z - H_1))/ \sinh(kH_1) \\ - \cosh(k(z + H_2))/ \sinh(kH_2) \\ ik/\omega \end{bmatrix} J_m(kr)e^{im\theta}$$

(2.12)

The number $A$ is arbitrary here and has the dimension of a hydrodynamic potential. The functions $J_m$ are Bessel-functions, $m \in \mathbb{Z}$ is the azimuthal wavenumber and $k$ is a radial wavenumber which can take any value. But to ensure that the boundary condition $u_r|_{r=R} = 0$ is satisfied, one must have

$$k = \kappa_{mn}/R, \quad J'_m(\kappa_{mn}) = 0,$$

(2.13)

where $\kappa_{mn}$ is the $n$-th zero of the derivative of the Bessel function $J_m$. The natural frequency $\omega$ of the (free) waves is

$$\omega = \pm \sqrt{\frac{(\rho_2 - \rho_1)gk + \gamma_1|2k^3}{\rho_2 \tanh^{-1}(kH_2) + \rho_1 \tanh^{-1}(kH_1)}}.$$

(2.14)

and may be either positive or negative for a fixed $m$. The solution presented here (see (2.10)) has a simple harmonic form, but the real flow is always real-valued. Taking the real part of (2.10), one finds rotating waves with positive rotation speed if $m\omega < 0$ (anti-clockwise when seen from above) and negative rotation speed if $m\omega > 0$ (clockwise when seen from above). Two oppositely rotating waves can also be superposed to give standing waves.

### 2.4. Sufficient conditions to apply perturbation methods

Perturbation methods may only be used when the Lorentz force and the viscous forces appear as small perturbations in the momentum balance. Symbolically, this means that

$$[J \times b_i], [j_i \times B], [\rho_i\nu_i \nabla^2 u_i] \ll [\rho_i\partial_t u_i], [\nabla p_i],$$

(2.15)

in terms of orders of magnitude (denoted using square brackets). This limits the parameter space that can be explored and we want to estimate these limits here.

For simplicity, we assume $J, B, \omega \geq 0$ to avoid the use of absolute values. We use $[x] = R$ as space scale rather than $k^{-1}$ since this is more explicit (and anyway similar to $k^{-1}$ for long wave-length waves). We denote $[u_i] = U$ the velocity scale which is arbitrary in this linear approach. For hydrodynamic waves, we can estimate

$$[\rho_i\partial_t u_i] \sim \rho_i\omega U, \quad [p_i] \sim \rho_i\omega RU.$$

(2.16)

The viscous term has the following order of magnitude:

$$[\rho_i\nu_i \nabla^2 u_i] \sim \rho_i\nu_i U/R^2.$$

(2.17)

We introduce $B$ as the order of magnitude of the imposed magnetic field

$$[B] \sim B = \begin{cases} \mu_0JR & \text{azimuthal} \\ \frac{B_z}{\mu_0JR} & \text{vertical}. \end{cases}$$

(2.18)

We can use two independent estimates related to either the azimuthal or to the vertical magnetic field. From Ohm’s law (2.3c), we then estimate the magnitude of the induction term as

$$[\sigma_i u_i \times B] \sim \sigma_i UB.$$

(2.19)

Still in Ohm’s law (2.3c), the electrical potential term may have the same order of
magnitude as \( u \times B \), but the interface deformations caused by the waves can also lead to a different magnitude which can be estimated from the boundary conditions (2.5a), (2.5b) and (2.7b). This yields the following two different estimates for the electrical current density

\[
[j_i] \sim [-\sigma_i \nabla \varphi_i] \sim \begin{cases} JU/(\omega R) & \text{interface deformations} \\ \sigma_i U B & \text{induction by the flow.} \end{cases}
\]

Using Ampère’s law (2.3d), we can calculate the associated magnetic field perturbation magnitudes

\[
[b_i] \sim \begin{cases} \mu_0 JU/\omega & \mu_0 \sigma_i U R B. \end{cases}
\]

All the orders of magnitude of the different fields have been estimated so we can now reformulate the inequalities (2.15). The trivial but sufficient condition to model viscous effects perturbatively is

\[
[\rho_i \nu_i \nabla^2 u_i] \ll [\rho_i \partial_t u_i] \iff Re_i = \frac{\omega R^2}{\nu_i} \gg 1,
\]

which we recognise as a constraint on Reynolds numbers based on the properties of the wave. A set of sufficient conditions that allow to model the effects of the Lorentz-force using perturbation methods is then

\[
\begin{align*}
[J \times b_i] &\ll [\rho_i \partial_t u_i] \iff \begin{cases} \frac{\mu_0 J^2}{\rho_i \omega^2} \ll 1 \\
\mu_0 \sigma_i J R B &\ll 1, \end{cases} \quad (2.23a) \\
[j_i \times B] &\ll [\rho_i \partial_t u_i] \iff \begin{cases} \frac{JB}{\rho_i \omega^2 R} \ll 1 \\
\sigma_i B^2 &\ll 1. \end{cases} \quad (2.23b)
\end{align*}
\]

Each time we have two estimates that are related to the two choices (2.20) that can be made. We will suppose that these a-priori sufficient conditions (2.22) and (2.23) are verified, but it is impossible at this stage to evaluate how necessary they are. A posteriori, when we have calculated the growth rates, we will be able to provide more reasonable estimates of the domain of validity of our perturbative model.

2.5. Simplifying assumptions in other studies on metal pad roll instability

Most existing theoretical works on metal pad roll instabilities have not been done with a perturbative approach, but they rely instead on many other simplifying assumptions. Here we discuss these assumptions and their impact to be able to situate how the present model is similar or different.

A first assumption frequently made is that the fluid layers are shallow. This assumption is relevant to model the long wavelength waves that occur in realistic Hall-Heroult cells; these waves are very wide (4–10 m in lateral extend) but very shallow (5–30 cm high). The shallow layer hypothesis is however not well adapted to the small laboratory scale reduction cells numerically studied in the present article and in Flueck et al. (2009); Steiner (2009), nor is it adapted to the experimental set-up of Pedchenko et al. (2009, 2016), or the metal pad roll studies in liquid metal batteries of Weber et al. (2017a, b).
We do not use the shallow approximation in our theory, but will expand our formulas in this limit in order to be able to compare our results to the existing models of Davidson & Lindsay \cite{DavidsonLindsay1998} and Lukyanov \textit{et al.} \cite{Lukyanovetal2001}.

A second frequent assumption made in the literature is that of inviscid fluids. This indeed simplifies the analysis, and since viscous effects are not crucial to the physical mechanism of the metal pad roll instability, the only ingredient that is missed is a dissipative mechanism. In realistic cells, far from the instability threshold, viscous effects are secondary, but in small cells near the instability threshold, one must account for the viscous damping in order to have a precise theory. We will provide a new explicit formula for the viscous damping rate of waves in cylindrical geometry.

A third typical assumption concerns Ohm’s law. An assumption very often adopted in metal pad roll theories consists of assuming that

\[
[\sigma_i u_i \times B] \ll [-\sigma_i \nabla \varphi_i] \implies \frac{\sigma_i \omega R B}{J} \ll 1,
\]

and when this inequality is satisfied, the dominant contribution to the electrical current perturbation comes from the interface motion rather than from the field induction in the bulk of the flow. In this limit, the magneto-static approximation

\[
j_i \approx -\sigma_i \nabla \varphi_i,
\]

can be used as a first order approximation of the quasi-static Ohm’s law \eqref{2.3a}. Physically, we then calculate the current density perturbation as if the metal slabs were solids at rest (even though they move). This hypothesis is quite well adapted to realistic Hall-Heroult cells, because in this case \(B = B_z\) always remains fairly low (\(\sim 1\) mT). Regarding the horizontal magnetic field \(B = \mu_0 J R\) though, it seems more easy to violate the assumption

\[
\sigma_i \mu_0 \omega R^2 \ll 1
\]

in large cells. The physical effect of the term \(\sigma_i u_i \times B_z e_z\) is well known: it creates a magnetic damping modelled in Sreenivasan \textit{et al.} \cite{Sreenivasanetal2005} for rectangular cells. The physical effect of the term \(\sigma_i u_i \times (\mu_0 J R / 2) e_\theta\) is less well described. It may be stabilizing but it can also be destabilizing as we know from studies on the Tayler instability \cite{Herremanetal2015}. Below we will model the effect of these quasi-static terms. We will provide an explicit formula for the magnetic damping due to the vertical magnetic field.

A fourth assumption frequently made is that

\[
[J \times b_i] \ll [j_i \times B] \iff \frac{\mu_0 J R}{B} \ll 1,
\]

which then leads to the following approximation of the Lorentz force:

\[
\text{Lorentz force}_i \approx j_i \times B.
\]

The induced field \(b_i\) does not need to be calculated, which is very practical as \(b_i\) is never easy to compute in finite fluid domains with realistic boundary conditions. Considering the definition \eqref{2.18} of \(b_i\), the condition \(\mu_0 J R / B \ll 1\) is only realizable when the vertical magnetic field dominates the horizontal field, which is rarely the case in industrial-size cells. The effect of the term \(J \times b_i\) was studied in Sun \textit{et al.} \cite{Sunetal2004}, and it was shown to have little impact on the instability. We will not ignore this term in the present paper. We will find that it mainly modifies the frequency of the waves, not their growth rates.

A fifth assumption usually adopted in the literature is that capillarity is not important.
For gravity waves, this means that

\[ kl_c \ll 1, \quad l_c = \sqrt{\frac{\gamma_1|2}{(\rho_2 - \rho_1)g}}. \] (2.29)

In other words, the capillary length \( l_c \) is assumed to be much smaller than the wavelength \( 2\pi/k \). This condition is very well verified by long-wavelength waves in industrial Hall-Heroult cells. This inequality does not hold in small cells; that is, in small cells the surface tension significantly modifies the frequency of the waves and the metal pad roll instability. Therefore, we will include the surface tension in our model as well.

Finally, a sixth simplifying assumption often made is that the electrical conductivity of both layers is very different:

\[ \frac{\sigma_1}{\sigma_2} \ll 1. \] (2.30)

This is more a fact than an assumption in any type of Hall-Heroult cells. Combined with the shallow nature of the cells and the electrical top and bottom boundary conditions, it implies that currents are mainly vertical in the electrolyte and mainly horizontal in the aluminium. Another consequence of this assumption is that the growth rates of the metal pad roll instability do not depend on the electrical conductivity of the layers, although the jump in conductivity is essential for this instability to occur. In this article, we will keep both \( \sigma_1 \) and \( \sigma_2 \) in our growth rate formula. This will allow us to estimate to which extend one can relax the conductivities of both layers in the direct numerical simulations as weaker conductivity jumps are more easily manageable for multiphase MHD codes.

### 2.6. Perturbative calculation of growth rates and frequency shifts

In this technical section, we calculate the theoretical expression for the complex eigenvalue \( s \) that characterizes the exponential behaviour (\( \sim \exp(st) \)) of the linear eigenmode. Since the perturbation methods that we are going to use are rather technical, we first give a brief overview of the methodology. The theoretical expression for \( s \) will be split as

\[ s = i\omega + (\lambda + i\delta)_\alpha. \] (2.31)

Here \( \omega \) is the inviscid gravity wave frequency and \( \alpha \) is a small complex shift induced by the Lorentz and the viscous forces. We define \( \lambda = \text{Re}(\alpha) \), the growth rate of the gravity wave, and \( \delta = \text{Im}(\alpha) \), the frequency shift. The growth rate is decomposed into several independent terms as follows:

\[ \lambda = \lambda_{v}^{\text{destab.}} + \lambda_{vv}^{\text{quasi-static}} + \lambda_{hh}^{\text{quasi-static}} + \lambda_{\text{visc. damping}}. \] (2.32)

The first term \( \lambda_v \leq 0 \) is the potentially destabilizing term. It is related to the Lorentz force: it comes from the interaction of the electrical current perturbation \( j_i \), caused by the interface deformation, with the background vertical magnetic field \( B_z \) (hence the suffix \( v \)). The terms \( \lambda_{vv}, \lambda_{hh} \) are quasi-static MHD corrections. The term \( \lambda_{vv} < 0 \) is always negative and is related to magnetic damping caused by \( B_z \). The term \( \lambda_{hh} \) is related to the azimuthal field; it may be slightly positive, but it is most often negative and is always very small. These quasi-static effects are rarely modelled in metal pad roll theories since they are quadratic in the magnetic field strength. The term \( \lambda_{\text{visc}} \leq 0 \) captures the viscous damping of the gravity waves. We similarly decompose the frequency shift as follows:

\[ \delta = \delta_h + \delta_{vh} + \delta_{\text{visc}}. \] (2.33)
The term \( \delta_h \) comes from the Lorentz force; it is the result of the interaction of \( \bm{j}_i \), caused by the interface deformation, with the background horizontal (azimuthal) magnetic field \( B_\theta = \mu_0 Jr/2 \) (hence the suffix \( h \)). The term \( \delta_h \) is a small quasi-static correction related to both \( B_\theta \) and \( B_z \), and \( \delta_{\text{visc}} \) is a frequency shift caused by viscous effects.

### 2.6.1. MPR instability in the magneto-static and inviscid limit: \( \lambda_v \& \delta_h \)

In this section, we focus on the leading terms in the growth rate and frequency shift formulas, \( \lambda_v \) and \( \delta_h \). This can be done in the inviscid limit \( \nu_i = 0 \), and, assuming that \( \sigma_i \omega R B/J \ll 1 \), the magnetostatic approximation of Ohm’s law \( \bm{j}_i \approx -\sigma_i \nabla \varphi_i \) can be used. The viscous and magnetic dissipations are ignored in this entire section.

We propose the following ansatz to the linear inviscid and magneto-static limit of the linearized problem defined in section 2.2:

\[
[u_i, p_i, \eta, \hat{\bm{j}}_i, \varphi_i, \hat{\bm{b}}_i] = \left( \left[ \hat{u}_i, \hat{\rho}_i, \hat{\eta}, \hat{\jmath}_i, \hat{\varphi}_i, \hat{\bm{b}}_i \right] \right) e^{i\omega t} e^{\alpha t}. \quad (2.34)
\]

In this notation, the tilded variables are small perturbations with respect to the hatted variables, and the complex shift \( \alpha \) is small with respect to \( \omega \). Here \( \omega \) is the frequency of the gravity waves. We inject this ansatz into the governing equations and we treat the Lorentz force term perturbatively. At leading order, we find the following balance equations:

\[
\begin{align*}
\rho_i (i\omega) \hat{u}_i + \nabla \cdot \hat{\rho}_i &= 0, \quad \nabla \cdot \hat{\bm{u}}_i = 0, \\
\hat{\jmath}_i &= -\sigma_i \nabla \hat{\varphi}_i, \quad \nabla \cdot \hat{\jmath}_i = 0, \\
\nabla \times \hat{\bm{b}}_i &= \mu_0 \hat{\jmath}_i, \quad \nabla \cdot \hat{\bm{b}}_i = 0. \quad (2.35)
\end{align*}
\]

Notice that the Lorentz force is absent in the momentum equation. The boundary conditions for the hatted fields can be copied from (2.4 \( \rightarrow \) 2.8) after replacing \( \partial_t \rightarrow i\omega \). We have to solve this problem in order to specify all the first order quantities. The hydrodynamical fields \( \hat{u}_i, \hat{\rho}_i, \hat{\eta} \) and the frequency \( \omega \) have already been given above. The electrical potential \( \hat{\varphi}_i \) is harmonic (\( \nabla^2 \hat{\varphi}_i = 0 \)) and is related to the surface elevation \( \hat{\eta} \) by the jump condition (2.7b). After some computations we find that

\[
\begin{bmatrix}
\hat{\varphi}_1 \\
\hat{\varphi}_2
\end{bmatrix} = \frac{JkA}{\omega} \begin{bmatrix}
\sigma_1^{-1} - \sigma_2^{-1} \\
\sigma_1^{-1} \tanh(kH_1) + \sigma_2^{-1} \tanh^{-1}(kH_2)
\end{bmatrix} \times 
\begin{bmatrix}
-i \sinh(k(z - H_1))/(\sigma_1 \cosh(kH_1)) \\
-i \cosh(k(z + H_2))/(\sigma_2 \sinh(kH_2))
\end{bmatrix}
J_m(\kappa r) e^{\text{i} m\theta}. \quad (2.36)
\]

The insulating sidewall condition is automatically satisfied since \( J_m'(kR) = 0 \) by definition (see (2.13)). The electrical top and bottom boundary conditions are satisfied as well. The electrical currents, \( \hat{\jmath}_i = -\sigma_i \nabla \hat{\varphi}_i \) are easily deduced from the potentials. The magnetic field perturbations \( \hat{\bm{b}}_i \) can be calculated by using for example the Biot-Savart law. This computation is very technical, but it is not necessary here: \( \hat{\bm{b}}_i \) will not be explicitly required to evaluate the complex shift \( \alpha \).

It is instructive to notice the systematic quadratures between the different leading order fields:

\[
\begin{align*}
\{ \varphi_1, \hat{u}_{i,r}, \hat{u}_{i,z}, \hat{\jmath}_{i,\theta}, \hat{\bm{b}}_{i,r}, \hat{\bm{b}}_{i,z} \} &\in \mathbb{R}, \\
\{ \hat{\eta}, \hat{u}_{i,\theta}, \varphi_i, \hat{\jmath}_{i,r}, \hat{\jmath}_{i,z}, \hat{\bm{b}}_{i,\theta} \} &\in \text{i}\mathbb{R}. \quad (2.37)
\end{align*}
\]

As shown below, this information alone allows us to understand why vertical magnetic fields are destabilizing and why horizontal magnetic fields can only induce a frequency shift in the perturbative limit.
All the leading order fields that are needed for the calculation of the shift \( \alpha \) are now specified. We are now ready to write the next order in the perturbation problem. From the balance of momentum and mass, we have

\[
\alpha \rho_i \tilde{u}_i + i \omega \rho_i \tilde{u}_i + \nabla \tilde{p}_i = \tilde{j}_i \times \tilde{B} + \tilde{J} \times \tilde{b}_i \tag{2.38a}
\]

\[
\nabla \cdot \tilde{u}_i = 0. \tag{2.38b}
\]

Note that the (small) Lorentz force is now present on the right-hand side of the momentum balance equation, and it can be calculated by using the first order fields \( \tilde{u}_i \) and \( \tilde{b}_i \). We are not considering the corrections \( \tilde{j}_i, \tilde{\varphi}_i, \tilde{b}_i \) at this stage as they do not affect the eigenvalue shift \( \alpha \). The hydrodynamical boundary conditions for the tilded variables can be copied from (2.4) and (2.5) except for the kinematic boundary conditions which now reads

\[
\alpha \tilde{\eta} + i \omega \tilde{\eta} = \tilde{u}_{i,z} \big|_{z=0}. \tag{2.39}
\]

Next to \( i \omega \tilde{\eta} \), we find the term \( \alpha \tilde{\eta} \) that is proportional to the small shift \( \alpha \). To find an equation for \( \alpha \), we express the solvability condition (also known as the Fredholm alternative). This condition is obtained by summing over the two fluid regions, \( i = 1, 2 \), the linear combination \( \int_{V_i} [\tilde{u}_i^* \cdot (2.38a) + \tilde{p}_i^* (2.38b)] dV \). We obtain

\[
\sum_{i=1,2} \int_{V_i} [\tilde{u}_i^* \cdot (\alpha \rho_i \tilde{u}_i + i \omega \rho_i \tilde{u}_i + \nabla \tilde{p}_i) + \tilde{p}_i^* \nabla \cdot \tilde{u}_i] \, dV = \sum_{i=1,2} \int_{V_i} \tilde{u}_i^* \cdot (\tilde{j}_i \times \tilde{B} + \tilde{J} \times \tilde{b}_i) \, dV. \tag{2.40}
\]

We introduce the notation \( \mathcal{P} \) for the “power” injected by the Lorentz-force. On the left-hand side, we use integration by parts to bring all the partial derivatives in space on the hatted variables. Using the leading order balances (2.35), we then find that

\[
\alpha \sum_{i=1}^2 \int_{V_i} \rho_i |\tilde{u}_i|^2 \, dV + \sum_{i=1}^2 \int_{\delta V_i} (\tilde{p}_i^* \tilde{u}_i + \tilde{p}_i \tilde{u}_i^*) \cdot n_i \, dS = \mathcal{P}. \tag{2.41}
\]

The first group of terms \( T_1 \) is proportional to kinetic energy. Using the boundary conditions for the hatted and tilded variables, we reduce the second group \( T_2 \) to

\[
T_2 = \alpha \int_{\delta S} (\tilde{p}_2^* - \tilde{p}_1^*) \big|_{z=0} \tilde{\eta} \, dS = \alpha \int_{\delta S} (-\gamma_{1|2} \nabla^2 \tilde{\eta}^* + (\rho_2 - \rho_1)g \tilde{\eta}^*) \, \tilde{\eta} \, dS
\]

\[
= \alpha \int_{\delta S} (\gamma_{1|2} |\nabla \tilde{\eta}|^2 + (\rho_2 - \rho_1)g |\tilde{\eta}|^2) \big|_{z=0} \, dS
\]

\[
= \alpha [(\rho_2 - \rho_1)g + \gamma_{1|2} \kappa^2] \int_{\delta S} |\tilde{\eta}|^2 \, dS. \tag{2.42}
\]

Notice that in the integration by parts we have used the boundary condition \( \partial_r \tilde{\eta} \big|_{r=R} = 0 \), which is always satisfied by the inviscid gravity waves if the contact angle is \( 90^\circ \). This reformulation allows us to see that \( T_2 \) is proportional to the potential energy. The quantity \( \kappa \) is real; it will be useful in what follows, and it can be explicitly calculated by using the expression for \( \tilde{\eta} \) (see Appendix A). In summary, \( T_1 + T_2 \) is proportional to the mechanical energy. Using again integration by parts, the fact that the waves derive from
harmonic potentials, see (2.11), and the hydrodynamic boundary conditions, one can further demonstrate that $T_1 = T_2$, for the simple harmonic wave under study:

$$T_1 = \alpha \left[ \rho_1 \oint_{\delta V_1} \hat{\phi}_1^* \nabla \hat{\phi}_1 \cdot \mathbf{n}_1 \, dS + \rho_2 \oint_{\delta V_2} \hat{\phi}_2^* \nabla \hat{\phi}_2 \cdot \mathbf{n}_2 \, dS \right]$$

$$= \alpha \int_S (\omega^* \hat{\phi}_1^* + i \omega \hat{\phi}_2^*) \, \eta \, dS$$

$$= \alpha \int_S (\hat{\phi}_2^* - \hat{\phi}_1^*) \, \eta \, dS = T_2.$$  \hspace{1cm} (2.43)

The immediate consequence of this identity is that the solvability condition (2.40) reduces to $2\alpha \mathcal{K} = \mathcal{P}$. Then the equation for the eigenvalue shift is $\alpha = \mathcal{P}/2\mathcal{K}$. Let us introduce the split $\mathcal{P} = \mathcal{P}_v + \mathcal{P}_h$ with

$$\mathcal{P}_v = \sum_{i=1,2} \int_{V_i} \hat{u}_i^* \cdot (j_i \times B_z e_z) \, dV, \hspace{1cm} (2.44a)$$

$$\mathcal{P}_h = \sum_{i=1,2} \int_{V_i} \hat{u}_i^* \cdot \left[ j_i \times (\mu_0 J_r/2) e_\theta + J e_z \times \hat{b}_i \right] \, dV, \hspace{1cm} (2.44b)$$

in order to clearly separate the contributions due to the vertical and the horizontal magnetic fields. In the formula for $\mathcal{P}_h$, the second term involves the magnetic field $\hat{b}_i$. Using the potential nature of the flow and $\nabla \times (J e_z) = 0$, we obtain

$$\hat{u}_i^* \cdot (J e_z \times \hat{b}_i) = \nabla \hat{\phi}_i^* \cdot (J e_z \times \hat{b}_i)$$

$$= \nabla \cdot \left( \hat{\phi}_i^* (J e_z \times \hat{b}_i) \right) + \hat{\phi}_i^* J e_z \cdot (\nabla \times \hat{b}_i).$$  \hspace{1cm} (2.45)

Using Ampère’s law $\nabla \times \hat{b}_i = \mu_0 j_i$ and after integrating over the volume this yields

$$\mathcal{P}_h = \sum_{i=1,2} \int_{V_i} \left\{ \hat{u}_i^* \cdot \left[ j_i \times (\mu_0 J_r/2) e_\theta \right] + \mu_0 J \hat{\phi}_i^* j_i \cdot e_z \right\} \, dV + \int_{\partial V} \hat{\phi}_i^* (\hat{b}_i \times \mathbf{n}_i) \cdot e_z \, dS.$$  \hspace{1cm} (2.46)

We see now why the use of the simplifying magnetic boundary condition (2.8) indeed allows us to calculate $\mathcal{P}_h$ without knowing $\hat{b}_i$. Note also that this assumption on the magnetic boundary condition only affects $\mathcal{P}_h$ and not $\mathcal{P}_v$.

Both $\mathcal{P}_v$ and $\mathcal{P}_h$ can be explicitly evaluated (see Appendix A), but without referring to these formulas, it is very instructive at this stage to see what we can learn from the quadrature stated in (2.37). A symbolic computation of the integrand in (2.44a) gives

$$\hat{u}_i^* \cdot (j_i \times B_z e_z) = \begin{vmatrix} \hat{u}_{r,i}^* & \hat{j}_{r,i} & 0 \\ \hat{u}_{\theta,i}^* & \hat{j}_{\theta,i} & 0 \\ \hat{u}_{z,i}^* & \hat{j}_{z,i} & B_z \end{vmatrix} \rightarrow \begin{vmatrix} \mathbb{R} & i\mathbb{R} & 0 \\ i\mathbb{R} & \mathbb{R} & 0 \\ \mathbb{R} & i\mathbb{R} & \mathbb{R} \end{vmatrix} \in \mathbb{R}. \hspace{1cm} (2.47)$$

Hence $\mathcal{P}_v \in \mathbb{R}$. Similarly we find that $\mathcal{P}_h \in i\mathbb{R}$. This implies that the eigenvalue shift $\alpha$ can be decomposed as follows:

$$\alpha = \frac{\mathcal{P}_v}{2\mathcal{K}} + \frac{\mathcal{P}_h}{2\mathcal{K}} \lambda_v. \hspace{1cm} (2.48)$$

This suggests an important physical fact. At this order of the expansion in our model, only the vertical magnetic field component $B_z$ can destabilize the gravity waves and
the corresponding growth rate is $\lambda_v$. The self-generated, horizontal magnetic field is not destabilizing but shifts the frequency of the wave by $\delta_h$. This result is coherent with the analysis of \cite{Sneyd1985} and \cite{SneydWang1994}.

A few pages of analytical calculations are necessary to evaluate all the integrals analytically, and we finally arrive at the following explicit formulas for the growth rate and frequency shift:

\begin{align}
\lambda_v &= \frac{\omega}{2} \frac{JB_z}{(\rho_2 - \rho_1)g + \gamma_1/2k^2} \frac{m}{(kR)^2 - m^2} \left( \tanh(kH_1) + \frac{kH_2}{\sinh^2(kH_2)} + \frac{1}{\tanh(kH_2)} \right) A \\
\delta_h &= \frac{\omega}{4} \frac{\mu_0 J^2 k^{-1}}{(\rho_2 - \rho_1)g + \gamma_1/2k^2} \frac{(kR)^2 - m^2}{(kR)^2 - m^2} \left( \frac{kH_1}{\sinh(kH_1) \cosh(kH_1)} \right) A, \tag{2.49b}
\end{align}

where

\begin{equation}
A = \frac{\sigma_1^{-1} - \sigma_2^{-1}}{\sigma_1^{-1} \tanh(kH_1) + \sigma_2^{-1} \tanh^{-1}(kH_2)}. \tag{2.49c}
\end{equation}

We have verified a posteriori the coherence of these formulas by comparing them with numerical evaluations of the integrals of $P_v$, $P_h$, $K$ computed using a simple quadrature rule on a two-dimensional $r - z$ grid.

In both $\lambda_v$ and $\delta_h$, we see dimensionless factors that balance the typical strength of the Lorentz force ($JB_z$ in $\lambda_v$, $\mu_0 J^2 k^{-1}$ in $\delta_h$) with respect to gravitational and capillary restoring forces ($((\rho_2 - \rho_1)g + \gamma_1/2k^2)$. We see that the growth rate $\lambda_v \sim m$, so axisymmetric waves with $m = 0$ can never be destabilized. The number $A$ is a conductivity-dependent factor that is not often seen in the metal pad literature. It clearly shows that a jump in electrical conductivity $\sigma_1 \neq \sigma_2$ is essential for instability.

We can derive an inviscid instability criterion for the case $\sigma_1 \ll \sigma_2$ commonly considered in the literature. Since $(kR)^2 - m^2 > 0$ always (see Appendix A), inviscid instability, that is $\lambda_v > 0$, requires

\begin{equation}
\text{Sgn}(JB_z m \omega) > 0 \quad \Rightarrow \quad \text{wave is MPR-unstable.} \tag{2.50}
\end{equation}

In reduction cells, we further have $J < 0$, as the reduction current flows from top to bottom, and if $B_z > 0$, this criterion tells us that only rotating waves with $m \omega < 0$ can be destabilized. This corresponds to waves rotating in the positive direction, or anticlockwise when seen from above, and this coincides with what can be expected from the Sele-mechanism \cite{Sele1977}. Note also that there is no threshold for the inviscid MPR-instability; this is coherent with previous results by \cite{DavidsonLindsay1998} and \cite{Lukyanovetal2001}.

The general formula for the growth rate takes simpler asymptotic forms in the limit of large conductivity jumps and for both deep fluid layers and shallow fluid layers. In the deep limit $kH_i \gg 1$ and with $\sigma_1/\sigma_2 \ll 1$ we find

\begin{align}
\lambda_{v,\text{deep}} &\approx \frac{JB_z}{(\rho_2 - \rho_1)g + \gamma_1/2k^2} \frac{m \omega_{\text{deep}}}{(kR)^2 - m^2}, \tag{2.51a} \\
\delta_{h,\text{deep}} &\approx 0, \tag{2.51b}
\end{align}

where

\begin{equation}
\omega_{\text{deep}} \approx \sqrt{\frac{(\rho_2 - \rho_1)gk + \gamma_1/2k^3}{\rho_1 + \rho_2}}. \tag{2.52}
\end{equation}
Metal pad roll instability in cylindrical reduction cells

For shallow layers \( kH_i \ll 1 \) and with \( \sigma_1/\sigma_2 \ll (kH_1)(kH_2) \) we have

\[
\lambda_{v,\text{shallow}} \approx \frac{JB_z}{(\rho_2 - \rho_1)g + \gamma_1^2 k^2} \frac{m\omega_{\text{shallow}}}{(kR)^2 - m^2 (kH_1)(kH_2)},
\]

(2.53a)

\[
\delta_{h,\text{shallow}} \approx \frac{\omega_{\text{shallow}}}{4} \frac{\mu_0 J^2 k^{-1}}{(\rho_2 - \rho_1)g + \gamma_1^2 k^2} \frac{(kR)^2 - 2m^2}{(kR)^2 - m^2} \frac{1}{kH_1},
\]

(2.53b)

where

\[
\omega_{\text{shallow}} \approx \sqrt{\frac{(\rho_2 - \rho_1)gk^2 + \gamma_1^2 k^4}{\rho_1 H_1^{-1} + \rho_2 H_2^{-1}}}
\]

(2.54)

The shallow approximation is commonly adopted in the metal pad roll literature. For long-wavelength waves, the surface tension is also usually ignored. The growth rate can be approximated in this limit as follows:

\[
\lambda_{v,\text{shallow}} \approx \frac{JB_z}{\sqrt{(\rho_2 - \rho_1)g(\rho_1 H_1^{-1} + \rho_2 H_2^{-1})}} \frac{R}{H_1 H_2} \frac{m}{\kappa_{mn}(\kappa_{mn} - m^2)}.
\]

(2.55)

In this formula, we have separated a wave-dependent factor that only depends on \( \kappa_{mn} \) and \( m \), from a factor that groups all other physical and geometrical parameters. This allows us to emphasize that the growth rate increases proportionally to the lateral extend \( R \) of the shallow cell (if \( J \) is held fixed). Alternatively, we can also write

\[
\lambda_{v,\text{shallow}} \approx \frac{m\omega_{\text{shallow}}}{\kappa_{mn}^2 - m^2 \kappa_{mn}^2} \beta, \quad \beta = \frac{JB_z}{(\rho_2 - \rho_1)g} \frac{R^2}{H_1 H_2}.
\]

(2.56)

The shallow-limit growth rate is proportional to the non-dimensional group \( \beta \), which appears in many previous studies on the metal pad roll instability.

In Appendix B, we develop further the small gap limit and show that the growth rate formula (2.56) is compatible with the existing shallow layer models. In the double limit of asymptotically small \( kH_i \ll 1 \) and \( \beta \ll 1 \), our model and those of Davidson & Lindsay (1998) and Lukyanov et al. (2001) yield precisely the same growth rates.

In Appendix C, we explore the effect on the metal pad roll instability of having symmetric boundary conditions at the top and the bottom of the cell (fixed potential or fixed normal current), as investigated in Munger & Vincent (2006). The mixed electrical boundary conditions we use in the present paper (fixed potential at the top and fixed normal current at the bottom) always yield in the shallow limit growth rates that are larger than when the boundary conditions are symmetric.

In deep cells, the growth rates depend very little on the nature of the boundary condition.

2.6.2. Quasi-static corrections: \( \lambda_{vv}, \lambda_{hh} \not\in \delta_{vh} \)

When the magnetic field strength \( B = \mu_0JR \) or \( B_z \) is large, the simplifying assumption \( \sigma_i \omega RB/J \ll 1 \) is no longer satisfied. In this case, the quasi-static approximation of MHD must include the retroaction of the fluid flow on the electrical current density through induction. The perturbations on the electrical current density must take the following form:

\[
j_i = \sigma_i(-\nabla \varphi_i + \mathbf{u}_i \times \mathbf{B}),
\]

(2.57)

instead of the static approximation \( j_i \approx -\sigma_i \nabla \varphi_i \) used in the previous section, at least in the aluminium layer where \( \sigma_2 \gg \sigma_1 \). The growth rate, \( \lambda_v \), and frequency shift, \( \delta_h \), are then augmented with the quasi-static corrections \( \lambda_{vv} \sim B_z^2, \lambda_{hh} \sim B^2 \) and \( \delta_{vh} \sim B \theta B_z \).
We modify the perturbation ansatz \((2.34)\) accordingly:

\[
[u_i, p_i, \eta, j_i, \varphi_i] = \left( \begin{array}{c}
\hat{u}_i, \hat{p}_i, \hat{\eta}, \hat{j}_i + \hat{F}^\nu_i + \hat{F}^h_i, \hat{\varphi}_i + \hat{J}^\nu_i + \hat{J}^h_i, \hat{b}_i + \hat{B}^\nu_i + \hat{B}^h_i
\end{array} \right) + \left( \begin{array}{c}
\hat{u}_i, \hat{p}_i, \hat{\eta}, \hat{j}_i, \hat{b}_i
\end{array} \right) e^{i\omega t} e^{\alpha t}.
\]  

(2.58)

The fields \(\hat{u}_i, \hat{p}_i, \hat{\eta}, \hat{j}_i, \hat{b}_i\) are identical to those defined in \((2.6.1)\) and capture all the magneto-hydrodynamical effects related to surface elevation. The quasi-static corrections for the electrical potential and the current densities are related to either the vertical (superscripts \(^v\)) or the horizontal magnetic field (superscript \(^h\)). That is to say, we define the current density corrections as follows:

\[
\hat{J}^\nu_i = \sigma_i (-\nabla \hat{\varphi}_i + \hat{u}_i \times B_z e_z),
\]

(2.59a)

\[
\hat{J}^h_i = \sigma_i (-\nabla \hat{\varphi}_i + \hat{u}_i \times (\mu_0 J r/2)e_\theta),
\]

(2.59b)

with the constraints: \(\nabla \cdot \hat{J}^\nu_i = \nabla \cdot \hat{J}^h_i = 0\). This in turn implies that electrical potentials satisfy the following Laplace and Poisson problems:

\[
\nabla^2 \hat{\varphi}_i^\nu = 0, \quad \nabla^2 \hat{\varphi}_i^h = -\mu_0 J \hat{u}_{z,i}
\]

(2.60)

with the following transmission and inhomogeneous boundary conditions:

\[
\hat{\varphi}_1^\nu \mid_{z=H_1} = 0, \quad \partial_z \hat{\varphi}_2^\nu \mid_{z=-H_2} = 0, \quad \partial_r \hat{\varphi}_1^\nu \mid_{r=R} = B_z \hat{u}_{i,\theta} \mid_{r=R},
\]

\[
\hat{\varphi}_1^\nu \mid_{z=0} = \hat{\varphi}_2^\nu \mid_{z=0}, \quad \sigma_1 \partial_z \hat{\varphi}_1^\nu \mid_{z=0} = \sigma_2 \partial_z \hat{\varphi}_2^\nu \mid_{z=0}
\]

(2.61)

and

\[
\hat{\varphi}_1^h \mid_{z=H_1} = 0, \quad \partial_z \hat{\varphi}_2^h \mid_{z=-H_2} = (\mu_0 J r/2) \hat{u}_{2,r} \mid_{z=-H_2},
\]

\[
\partial_r \hat{\varphi}_1^h \mid_{r=R} = -(\mu_0 J R/2) \hat{u}_{i,z} \mid_{r=R}, \quad \hat{\varphi}_1^h \mid_{z=0} = \hat{\varphi}_2^h \mid_{z=0},
\]

\[
\sigma_1 \left(-\partial_z \hat{\varphi}_1^h + (\mu_0 J r/2) \hat{u}_{1,r} \right) \mid_{z=0} = \sigma_2 \left(-\partial_z \hat{\varphi}_2^h + (\mu_0 J r/2) \hat{u}_{2,r} \right) \mid_{z=0}.
\]

(2.62)

Since the problems defined here have no simple analytical solution, we have estimated \(\hat{\varphi}_i^\nu\) and \(\hat{\varphi}_i^h\) numerically. We use a Fourier expansion in the azimuthal direction; that is, we set \(\hat{\varphi}_i^\nu = f_i^\nu(r, z) \exp(i m \theta), \hat{\varphi}_i^h = f_i^h(r, z) \exp(i m \theta)\), and we solve the two-dimensional problems for \(f_i^\nu\) and \(f_i^h\) using a standard second order finite difference method on a uniform grid with \(2 \times (M+1)^2\) points:

\[
\begin{cases}
\text{region 1:} & (i\delta r, j\delta z_1), \quad \forall i, j \in \{0, 1, \ldots, M\} \\
\text{region 2:} & (i\delta r, -j\delta z_2)
\end{cases}
\]

Here \(\delta r = R/M, \delta z_1 = H_1/M, \delta z_2 = H_2/M\). Radial and vertical derivatives appearing in the boundary conditions are discretised using second order uncentered finite difference formulas. At the axis, we enforce the regularity condition \(\hat{\varphi}_i^\nu \mid_{r=0} = \hat{\varphi}_i^h \mid_{r=0} = 0\) for \(m \neq 0\). The code is provided as supplementary material.

As in the previous section, we do not need to calculate the quasistatic magnetic field corrections provided that they satisfy the boundary condition \((2.8)\).

Without any numerical calculation, we observe that the following quadratures hold:

\[
\begin{align*}
\{\hat{\psi}_i^h, \hat{F}_{i,r}^h, \hat{F}_{i,z}^h, \hat{J}_{i,\theta}^h, \hat{B}_{i,r}^h, \hat{B}_{i,z}^h, \hat{B}_{i,\theta}^h\} & \in \mathbb{R} \\
\{\hat{\psi}_i^\nu, \hat{F}_{i,r}^\nu, \hat{F}_{i,z}^\nu, \hat{J}_{i,\theta}^\nu, \hat{B}_{i,r}^\nu, \hat{B}_{i,z}^\nu, \hat{B}_{i,\theta}^\nu\} & \in i\mathbb{R}.
\end{align*}
\]

(2.63)
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Supposing that the quasi-static field corrections are known, we recalculate the complex eigenvalue shift $\alpha$ by writing a modified solvability condition. This leads here to

$$2\alpha K = \mathcal{P} + \sum_{i=1,2} \int_{V_i} \hat{u}_i^* \cdot \left[ \left( \mathcal{F}_i^V + \mathcal{F}_i^h \right) \times B + J \times \left( \mathcal{B}_i^V + \mathcal{B}_i^h \right) \right] \, dV. \tag{2.64}$$

The quantities $K$ and $\mathcal{P}$ are as before. The new term $\mathcal{Q}$ isolates the effect of the quasistatic field corrections. We divide it into three parts

$$\mathcal{Q} = \mathcal{Q}_{vv} + \mathcal{Q}_{hh} + \mathcal{Q}_{vh}, \tag{2.65}$$

with

$$\mathcal{Q}_{vv} = \sum_{i=1,2} \int_{V_i} \hat{u}_i^* \cdot \left( \mathcal{F}_i^V \times B_z e_z \right) \, dV, \tag{2.66}$$

$$\mathcal{Q}_{hh} = \sum_{i=1,2} \int_{V_i} \hat{u}_i^* \cdot \left( \mathcal{F}_i^h \times (\mu_0 J r/2) e_\theta + J \times \mathcal{B}_i^h \right) \, dV, \tag{2.67}$$

$$\mathcal{Q}_{vh} = \sum_{i=1,2} \int_{V_i} \hat{u}_i^* \cdot \left( \mathcal{F}_i^h \times B_z e_z + \mathcal{F}_i^V \times (\mu_0 J r/2) e_\theta + J \times \mathcal{B}_i^V \right) \, dV. \tag{2.68}$$

Using the quadrature properties (2.63), we readily deduce that $\mathcal{Q}_{vv}, \mathcal{Q}_{hh} \in \mathbb{R}$ whereas $\mathcal{Q}_{vh} \in i\mathbb{R}$. With $\alpha = \lambda + i\delta$, we obtain the following modified formulas for the growth rate $\lambda$ and frequency shift $\delta$:

$$\lambda = \lambda_v + \frac{\mathcal{Q}_{vv}}{2K \lambda_v}, \quad \delta = \delta_h + \text{Im} \left( \frac{\mathcal{Q}_{vh}}{2K \delta_h} \right). \tag{2.69}$$

In practice, the integrals in $\mathcal{Q}_{vv}$, $\mathcal{Q}_{hh}$, $\mathcal{Q}_{vh}$ are calculated numerically using the field profiles provided by the finite difference code. We use a simple trapezoidal quadrature rule to calculate the two-dimensional integral over $(r, z)$. The terms involving the magnetic field corrections in $\mathcal{Q}_{hh}$ and $\mathcal{Q}_{vh}$ are transformed as explained in (2.45) and (2.46).

The need to use a numerical approach to find the quasi-static corrections is slightly unsatisfying. We now consider an alternative approach using the fact that $\sigma_1 \ll \sigma_2$. In this case, the induction in the top electrolyte layer is very weak, and we can use the approximation

$$\mathcal{F}_1^V \approx \mathcal{F}_1^h \approx 0 \tag{2.70}$$

in the electrolyte. We then need to solve (2.60) in the aluminium layer only, with the boundary conditions stated in (2.61), (2.62), and the transmission conditions replaced by

$$\partial_z \hat{\psi}_2^V |_{z=0} \approx 0, \quad \partial_z \hat{\psi}_2^h |_{z=0} \approx (\mu_0 J r/2) \hat{u}_{2,r} |_{z=0}. \tag{2.71}$$

Analytical solutions can be obtained in terms of series of harmonic functions. The calculation for $\hat{\psi}_2^h$ is quite tedious and is not presented here for brevity. The computation for $\hat{\psi}_2^V$ is easier and we obtain

$$\hat{\psi}_2^V = \left[ C_0 \left( \frac{r}{R} \right)^m + \sum_{j=1}^{+\infty} C_j I_m(j\pi r/H_2) \cos(j\pi z/H_2) \right] e^{im\theta}. \tag{2.72}$$

Here $I_m$ is a modified Bessel function. The coefficients $C_0, C_1, C_2, \ldots$ are calculated by
projecting the inhomogeneous boundary condition $\partial_r \hat{\Psi}_2^r |_{r=R} = \widehat{u}_\theta, 2 |_{r=R} B_z$ onto the set of the basis functions \{$\cos(j \pi z / H_2), \ j \in \mathbb{N}$\} using the orthogonality relation

$$\int_{-H_2}^{0} \cos \frac{j \pi z}{H_2} \cos \frac{j' \pi z}{H_2} \, \delta_{j,j'} \left\{ \begin{array}{l l} H_2, & \ j = 0 \vspace{0.5em} \\ H_2 / 2, & \ j \neq 0. \end{array} \right.$$  

This yields

$$C_0 = -\imath \frac{AB_z}{k H_2} J_m(kR),$$  

$$C_j = -\imath \frac{AB_z}{j \pi} \frac{4mJ_m(kR)}{(kR)(I_{m+1}(j \pi R / H_2) + I_{m-1}(j \pi R / H_2))} \frac{k^2}{k^2 + (j \pi / H_2)^2} \sinh(kH_2).$$

With this information we can calculate the integral $Q_{vv}$, using the following expression:

$$Q_{vv} \approx -\sigma_2 \int_{V_2} \| \hat{u}_i^* \times B_z e_z \|^2 \, dV + \int_{V_2} \hat{u}_i^* \cdot \left( -\sigma_2 \nabla \hat{\Psi}_2^r \times B_z e_z \right) \, dV.$$  

Explicit evaluation of all the integrals yields the following formula for the magnetic damping rate $\lambda_{vv} \approx Q_{vv} / 2K$:

$$\lambda_{vv} \approx \frac{\sigma_2 B_z^2}{\rho_1 \tanh(kH_1) + \rho_2 \tanh(kH_2)} \left\{ -\frac{1}{4} \left( \frac{1}{\tanh(kH_2)} + \frac{kH_2}{\sinh^2(kH_2)} \right) + \left( \frac{1}{k^2 R^2 - m^2} \right) \times \left[ \frac{m}{kH_2} + 4m^2 \sum_{j=1}^{+\infty} \left( \frac{1}{j \pi} \frac{k^3}{R(k^2 + (j \pi / H_2)^2)} \frac{I_{m+1}(j \pi R / H_2)}{I_{m-1}(j \pi R / H_2)} \right) \right] \right\}.$$  

The first term in the curly brackets is negative; this would be the only dampening term in containers with perfectly conducting walls (where $\hat{\Psi}_2^r = 0$). The collection of terms on the second line is positive; these terms related to the electrically insulating radial boundaries make the magnetic damping less efficient. This physical observation is coherent with Sreenivasan et al. (2005). In practice, we observe a fast exponential decay of the terms in the sum over $j$: the shallower the cell, the faster the decay. All the computations reported in the paper are done by truncating the sum over $j$ to the first 50 terms. We have verified the validity of the truncated formula by comparing it with the result from the finite difference code (demonstrated below).

Expanding the previous formula for the magnetic damping rate in the shallow limit, $kH_i \ll 1$ gives

$$\lambda_{vv, shallow} \approx \frac{\sigma_2 B_z^2}{\rho_1 \left( \frac{H_2}{H_1} \right) + \rho_2} \left( -\frac{1}{2} + \frac{m}{k^2 R^2 - m^2} \right).$$  

No significant simplification occurs in (2.76) in the deep limit $kH_i \gg 1$. This is due to the fact that the cosine basis used in the expansion is not adapted in this case.

### 2.6.3. Viscous corrections: damping $\lambda_visc$, frequency shift $\delta_visc$

We finally include the viscous effects in the perturbative calculation. The leading order of the viscous effects can be estimated by assuming that the cell is in a purely hydrodynamical state $J = B_z = 0$, which is then a rather classical problem.

Early calculations of viscous damping rates of waves in cylinders go back to Case & Parkinson (1957) (only one fluid ($\rho_1 = 0$)). The theory compares well to experimental
measurements in well polished cylinders, but also that it is certainly not adequate for all liquids in all vessels. Discrepancies are caused by the fact that the viscous damping may significantly depend on the complex contact line dynamics (contact-angle hysteresis) and on surface contamination (see discussions in e.g. [Case & Parkinson (1957); Miles & Henderson (1990)]). This makes the theoretical calculation of surface wave damping rates delicate.

Ignoring contact line and contamination issues, we still have the problem that viscous damping rate formulas for waves in cylinders filled with two layers of fluid are not available in the literature. Giving an answer to this question is the purpose of the present section and the appendix D. All the details of our calculations are provided in the appendix D.

The method we use is different from that of [Case & Parkinson (1957)], but it fits better the asymptotic framework we have been using in the previous sections. In a nutshell, we further extend the asymptotic ansatz (2.58) for the velocity and the pressure as follows:

\[
[u_i + \bar{u}_i, p_i + \bar{p}_i, \ldots] = \left( [\hat{u}_i + \hat{u}_i', \hat{p}_i + \hat{p}_i', \ldots] + [\bar{u}_i + \bar{u}_i', \bar{p}_i + \bar{p}_i', \ldots] \right) e^{i(\omega + \alpha)t}. \tag{2.78}
\]

The fields \(\hat{u}_i, \hat{p}_i\) are the inviscid gravity wave profiles. Barred variables designate boundary layer corrections that only exist in \(O(\sqrt{\nu_i/\omega})\) wide layers near the rigid walls. We calculate these viscous boundary layers by ignoring the specificity of the contact line region and by ignoring contributions from the free interface, which supposably are weaker (not always justifiable for arbitrary fluids). Mass conservation in the viscous boundary layer causes a wall-normal pumping flow that modifies the solvability condition (2.40) via boundary terms. After discarding unphysical terms originating from the bad modeling of the contact line region, we find

\[
\lambda_{\text{visc}} = -\frac{1}{\sqrt{2}} \sum_{i=1, 2} \frac{\sqrt{\nu_i |\omega|}}{R^2} \rho_i \left[ k(R - H_i) \sinh^{-2}(kH_i) + \left( \frac{k^2 R^2 + m^2}{k^2 R^2 - m^2} \right) \tanh^{-1}(kH_i) \right] \rho_1 \tanh^{-1}(kH_1) + \rho_2 \tanh^{-1}(kH_2),
\]

and the corresponding frequency shift is

\[
\delta_{\text{visc}} = \text{Sgn}(\omega) \lambda_{\text{visc}}. \tag{2.80}
\]

This shift always decreases the absolute value of the frequency of the wave.

In appendix D we further show that the above damping rate \(\lambda_{\text{visc}}\) is identical to the one obtained by using the energy method suggested in [Case & Parkinson (1957)].

In the shallow limit \(kH_i \ll 1\), damping on the bottom and the top boundaries dominates, and in this case we can derive the asymptotic form

\[
\lambda_{\text{visc, shallow}} \approx -\frac{1}{\sqrt{2}} \rho_1 H_1^{-2} \sqrt{\nu_1} + \rho_2 H_2^{-2} \sqrt{\nu_2} \frac{1}{\rho_1 H_1^{-1} + \rho_2 H_2^{-1}} \sqrt{|\omega_{\text{shallow}}|.} \tag{2.81}
\]

In the deep limit, \(kH_i \gg 1\), damping on the lateral wall dominates, and in this case the following approximation holds:

\[
\lambda_{\text{visc, deep}} \approx -\frac{1}{\sqrt{2}} \sqrt{\left( \frac{k^2 R^2 + m^2}{k^2 R^2 - m^2} \right) \frac{1}{R}} \frac{\rho_1 \sqrt{\nu_1} + \rho_2 \sqrt{\nu_2}}{\rho_1 + \rho_2} \sqrt{|\omega_{\text{deep}}|.} \tag{2.82}
\]
2.7. Applicability of the perturbative model: necessary conditions

We have provided sufficient conditions to apply the perturbation method, but necessary conditions can be derived a posteriori by verifying that

\[
\frac{\lambda_v}{\omega} \ll 1, \quad \frac{\lambda_{vv}}{\omega} \ll 1, \quad \frac{\lambda_{hh}}{\omega} \ll 1,
\]

\[
\frac{\lambda_{\text{visc}}}{\omega} \ll 1, \quad \frac{\delta_h}{\omega} \ll 1, \quad \frac{\delta_{vh}}{\omega} \ll 1.
\]

(2.83)

When these conditions are not met, the theoretical formulas we have deduced in the above sections may not be accurate or may even be wrong. This observation must be kept in mind when applying our theoretical formulas.

3. Experimental study of viscous damping rates

In [Horstmann et al. (2018b)], a detailed experimental study on orbital sloshing of waves in two and three fluid layer systems has been performed. In this configuration, the cylinder is horizontally driven along a circular path. This motion causes a large scale rotating wave \((m,n) = (1,1)\), which is most frequently associated with the metal pad roll instability. After suddenly stopping the orbital shaker we measure the viscous decay of this rotating gravity wave and compare it to the theoretical formula (2.79).

The experimental set-up is shown in figure 2 (a). Various cylindrical containers of different aspect ratios have been made from polished acrylic glass. For the present study we use containers with radius \(R = 5\) cm and total height \(H_1 + H_2 = 10\) cm with \(H_2\) varying in the range \([0.15\text{ cm}, 8.5\text{ cm}]\) in steps of 0.5 cm. Both the top and bottom caps of the cylinder have ten supply holes uniformly distributed on a circle at 42 mm from the vertical axis. These holes are used to attach up to ten ultrasound probes. A small layer of acrylic glass 5 mm thick is kept between the probes and the observation volume to ensure non-invasive ultrasound measurements of the interfacial motion. The upper cap additionally contains a small filling hole of diameter 4 mm to degas the cell and to adjust the position of the interface. The cylinder is placed on a sample holder to facilitate ultrasound measurements also from below. This way, up to 20 ultrasound probes can be simultaneously employed making it possible to precisely reconstruct wave modes. The measurements have been done with the Ultrasound Doppler Velocimeter DOP 3010 from Signal-Processing. By directly identifying and tracking the ultrasound echo reflected by the interface, the interfacial wave motion can be precisely measured for amplitudes in the range \([0.15\text{ mm}, 10\text{ mm}]\). This novel acoustic measurement method can also be used with opaque liquid metals. Finally, the sample holder is mounted onto a Kuhner LS-X lab-shaking table \((420 \times 420\text{ mm})\), which can prescribe ideal circular motions to the vessel parameterized as follows:

\[
r_{\text{orbital}}(t) = \frac{d_s}{2} (\cos(\Omega t) e_x + \sin(\Omega t) e_y).
\]

(3.1)

Here \(\Omega\) is a constant frequency and \(d_s\) is the diameter of the circular motion of the vertical axis of the vessel. The shaker is designed to allow for a continuous adjustment of the shaking diameter up to \(d_s = 70\) mm and to set shaking frequencies from \(\Omega = 20\text{ min}^{-1}\) up to \(\Omega = 500\text{ min}^{-1}\) in steps of 1 min\(^{-1}\).

Different working liquids were tested in [Horstmann et al. (2018b)] in order to realize long-time stable interfaces in two- and three-layer systems with distinct wave properties and boundary conditions. A special combination of fluids has been prepared for the present study. The top layer is composed of Paraffinum Perliquidum \((\rho_1 = 0.846\text{ g/cm}^3\text{,})\)
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(a) Experimental set-up used to measure viscous wave damping

![Experimental set-up](image)

(b) Measured and theoretical viscous damping rates

Figure 2. (a) Picture of the experimental set-up showing the cylindrical vessel mounted on a Kuhner LS-X lab-shaker and different parts composing vessels of various aspect ratios. In this picture, the cylinder is filled with silicon oil (top), water (middle), and paraffin oil (bottom) to realize a stable three-layer system; the water layer is coloured to increase visibility. (b) Measured damping rates $\lambda$ are compared with the theoretical viscous damping rates from (2.79) for different interfacial heights $H_2$. The theoretical damping rates in the shallow limit (2.81) and the theoretical damping rates in the deep limit (2.82) are also shown.

We used the following strict protocol to measure the experimental damping rates. Pre-measurements were conducted to find the eigenfrequency $\Omega$ of the mode $(m, n) = (1, 1)$ for each choice of $H_2$, since it is known from [Reclari et al. (2014)] that at resonance the
forced wave motion is close to that of free gravity waves. We then drive the rotating wave at the resonant frequency, and adjust the shaking diameter $d_s$ so that the wave is large enough to allow measurements of the damping rate but is small enough to remain in the linear, non-breaking wave regime (amplitudes between 3 mm and 5 mm in practice). The table is turned off after the wave has settled in a saturated state. Subsequently, we give the wave some time (about one up to two periods) to become a "free" gravity wave. Then the exponential decay rate $-\lambda$ is determined by fitting the measured wave-amplitude with the ansatz

$$A(t) = A_0 e^{\lambda t} \cos(\Omega t - \xi).$$  \hspace{1cm} (3.2)

Here $A_0$ denotes the initial wave amplitude which also needs to be fitted; this additional fit introduces some uncertainty. In figure 2(b) we compare the measured damping rates to the theoretical formula. Error bars were estimated by using different fitting time intervals. The measured decay rates are in very good agreement with the theoretical viscous damping rate, although the theory seems to underestimate slightly the measured damping rates. Figure 2(b) also shows the theoretical damping rates in the shallow and deep limits. The measurements coincide with the deep limit when $H_1 = H_2 = 5$ cm, and they coincide with the shallow limit when either $H_1 \ll H_2$ or $H_1 \gg H_2$. In conclusion we have compared the damping rates in the full transitional regime between these simplified asymptotic limits which are frequently analysed in the literature.

The full experimental study of orbital sloshing of two and three-layer fluid systems is presented in a separate article (Horstmann et al. 2018b). Here we conclude that our perturbation theory suitably predicts the viscous damping rates of gravity waves of two-layer systems when the simplified boundary conditions at the contact line are fulfilled.

4. Applications

We now return to the metal pad roll problem and apply our theory to two reduction cells: a small one and a large one. An on-scale sketch of these two cells is shown in Figure 3. We will also apply the theory to a liquid metal battery. In the entire section we use the constants $g = 9.81 \text{ms}^{-2}$ and $\mu_0 = 4\pi \times 10^{-7} \text{TmA}^{-1}$.

4.1. Small, laboratory scale cell

We start with a small scale reduction cell that has been numerically investigated in Flueck et al. (2009) and Steiner (2009). The dimensions of the cell are

$$R = 0.035 \text{m}, \quad (H_1, H_2) = (0.075 \text{m}, 0.075 \text{m}).$$  \hspace{1cm} (4.1)
different terms contributing to the frequency shift. The quantities $k$ the wavenumbers $\omega$ in $T$, $\gamma$ surface tension $\lambda$ are $\lambda_{visc}$, $\lambda_{hh}$, $\delta_{h}$ and $\delta_{vh}$ in $s^{-1}$.

The material properties of the cryolite and aluminium that we use are

\begin{align*}
(\rho_1, \rho_2) &= (2150, 2300) \text{kgm}^{-3}, \\
(\sigma_1, \sigma_2) &= (250, 3.5 \times 10^6) \text{Sm}^{-1}, \\
(\nu_1, \nu_2) &= (1.2, 0.52) \times 10^{-6} \text{m}^2\text{s}^{-1}.
\end{align*}

This cell bares very little resemblance with industrial Hall-Heroult cells but it is representative of a laboratory scale device such as those of Pedchenko et al. (2009, 2016). Small cells can only be destabilized with currents $J$ or vertical magnetic inductions $B_z$ that are large enough. We decide to keep $J$ within realistic levels, but we take the liberty to impose very high magnetic fields $B_z$, as done in the references mentioned above. As shown in Steiner (2009), surface tension may be important in small cells, so here we compare simulations without and with surface tension. We take $\gamma_{1|2} = 0.5 \text{Nm}^{-1}$ (value from Steiner (2009)).

4.1.1. Theoretical results

Table 1 provides the numerical values obtained from our theory (in SI units) for the wavenumbers $k$ and the frequencies $\omega$ of several waves ($m, n$) together with all the different terms $\lambda_{v}, \lambda_{vv}, \lambda_{hh}, \lambda_{visc}$ contributing to the growth rate and the terms $\delta_{h}, \delta_{vh}$ contributing to the frequency shift. The quantities $\lambda_{v}, \lambda_{vv}, \lambda_{hh}, \delta_{h}, \delta_{vh}$ are given relative to $J, J B_z, J^2$. No data for axisymmetric waves, $m = 0$, are reported since these waves can never become unstable according to our theory.

By inspecting the column $\lambda_{v}/J B_z$, we see that the large scale sloshing mode $(m, n) = (1,1)$ is not the only unstable wave. Without surface tension, the higher $m$ modes are
Figure 4. Small cell. Marginal stability curves in the $J-B_z$ plane for metal pad roll instability of waves $(m,n)$ without surface tension (a) or with surface tension (b). In the regions shaded in grey the cell is stable. Surface tension has stabilizing effects. The fundamental wave $(m,n) = (1,1)$ always becomes unstable first.

From the previous table, it is clear that all the effects induced by the horizontal fields are negligible for realistic values of $J \sim 10^4$ Am$^{-2}$. In the small cell geometry, we can therefore approximate the growth rate and frequency shift as follows:

$$\lambda \approx \lambda_v + \lambda_{vv} + \lambda_{visc}, \quad \delta \approx \delta_{visc}. \tag{4.3}$$

In Figure 4, we show curves of marginal instability in the $J-B_z$ plane calculated using the data of Table 1. The first mode that becomes unstable is the large scale sloshing wave, $(m,n) = (1,1)$. This mode is then followed by the other waves $(m,1)$ with $m \geq 2$. In absence of surface tension, the marginal stability curves for the different waves are very close to each other, so that many different waves may become unstable at the same time. When the surface tension is active, there is a clear preference for the $(1,1)$ mode; the high $m$ modes are less unstable. No waves with smaller radial structures $(n \geq 2)$ are unstable in the portion of the parameter space shown in the figure.

In Figure 5-(a), we show the currents $\hat{j}_i$, $\hat{J}_v$, and $\hat{J}_h$ that are associated with the
Fundamental wave \((1,1)\) for an equilibrium electrolysis current \(J\) flowing from the top to the bottom. The deformed interface is represented by the black line. The current lines \(\mathbf{j}_i\), due to the interface motion, loop in a symmetrical way in both fluid layers. The quasi-static corrections \(\mathcal{J}_v^i, \mathcal{J}_h^i\) on the other hand, are mainly confined inside the aluminium and remain away from the boundaries so that the approximatively insulating boundary conditions are satisfied. In panel (b) of Figure 5, we show the power densities

\[
\hat{u}_i^* \cdot (\mathbf{j}_i \times B_z \mathbf{e}_z), \quad \hat{u}_i^* \cdot (\mathcal{J}_v^i \times B_z \mathbf{e}_z), \quad \hat{u}_i^* \cdot (\mathcal{J}_h^i \times (\mu_0 J_r / 2 e_\theta)) + \mu_0 \hat{\phi}_i^* \hat{u}_{i,z},
\]

(4.4)

that appear in the integrands of \(P_v\), \(Q_{vv}\) and \(Q_{hh}\). This allows to visually locate where the instability is powered or inversely where magnetic damping takes place. In the small cell under consideration, we see that both the cryolite and the aluminium participate in a symmetric way to the destabilization through \(P_v\). The density of \(Q_{vv}\) is negative in the bottom aluminium layer, where the flow induction takes place. Although \(Q_{hh}\) is small in this small reduction cell, we notice that the two terms composing the density of \(Q_{hh}\) have the following properties:

\[
\hat{u}_i^* \cdot (\mathcal{J}_h^i \times (\mu_0 J_r / 2 e_\theta)) + \mu_0 \hat{\phi}_i^* \hat{u}_{i,z} < 0,
\]

\[
\mu_0 \hat{\phi}_i^* \hat{u}_{i,z} > 0.
\]

(4.5)

The first term is stabilizing \((< 0, \text{blue regions})\) and the second is destabilizing \((> 0, \text{red regions})\). These terms are not active at the same place in space.

In the theoretical section, we have provided exact formulas for the frequency \(\omega\) and the growth rate \(\lambda_v\) in (2.49). We have also derived deep and shallow limits. In Table 2 we apply these formulas to the small cell under consideration. We focus our attention on the two waves \((m,n) = (1,1), (2,1)\) and ignore the surface tension effects. The deep limit formula (2.51) clearly provides an excellent approximation, but the shallow-limit formulas (2.53) are not adapted here.

In Table 3 we compare the theoretical estimation of \(\lambda_{vv}/B_z^2\) using formula (2.76) to numerically calculated values. The numerical values are obtained using numerical quadrature and the field profiles \(\hat{\Psi}_v^i\) that result from the finite difference code (see discussion after (2.62)). Increasing the spatial resolution \(M = 100, 200, 400, 800\) in the code, we observe that the values of the damping rates converge towards the theoretical prediction of (2.76). This validates our finite difference solver, and also suggests that our theoretical formula (2.76) for the magnetic damping rate is correct.
Table 2. Comparisons between the theoretical formulas in (2.49) and the deep and shallow limits for inviscid frequencies \( \omega \) (in s\(^{-1}\)), relative growth rates \( \lambda_{v}/JB_{z} \) (in sm\(^{-2}\)kg\(^{-1}\)), relative damping term \( \lambda_{vv}/B_{z}^{2} \) (in s\(^{-1}\)T\(^{-2}\)), and viscous damping rates \( \lambda_{\text{visc}} \) (in s\(^{-1}\)). Surface tension is ignored. The deep limit is a good approximation in the small cell.

\[
\begin{array}{cccccc}
(m, n) & \text{model} & \omega & \lambda_{v}/JB_{z} & \lambda_{vv}/B_{z}^{2} & \lambda_{\text{visc}} \\
(1, 1) & \text{exact} & 4.169 & 1.189 \times 10^{-3} & -66.47 & -0.0684 \\
& \text{deep} & 4.171 & 1.185 \times 10^{-3} & -66.42 & -0.0245 \\
& \text{shallow} & 8.284 & 1.513 \times 10^{-4} & -98.00 & -0.0315 \\
(2, 1) & \text{exact} & 5.372 & 1.370 \times 10^{-3} & -78.75 & -0.1056 \\
& \text{deep} & 5.372 & 1.370 \times 10^{-3} & -78.75 & -0.1056 \\
& \text{shallow} & 13.740 & 8.184 \times 10^{-3} & -98.00 & -0.0315 \\
\end{array}
\]

Table 3. The relative magnetic damping rates \( \lambda_{vv}/B_{z}^{2} \) (in s\(^{-1}\)T\(^{-2}\)) are calculated numerically for different waves \((m, n)\) using a finite difference solver with variable spatial resolution \( M = 100, 200, 400, 800 \) and compared to the theoretical estimate (2.76) obtained by assuming \( \sigma_{1} \ll \sigma_{2} \). The numerical values converge towards the theoretical estimate.

\[
\begin{array}{cccccc}
(m, n) & \text{num (100)} & \text{num (200)} & \text{num (400)} & \text{num (800)} & \text{theoretical} \\
(1, 1) & -66.47 & -66.37 & -66.34 & -66.33 & -66.31 \\
(2, 1) & -79.08 & -78.82 & -78.76 & -78.75 & -78.71 \\
(3, 1) & -87.19 & -86.77 & -86.66 & -86.63 & -86.60 \\
\end{array}
\]

**Figure 6.** Variation of the relative growth rate \( \lambda_{v}/JB_{z} \) (panel (a)) and magnetic damping term \( \lambda_{vv}/B_{z}^{2} \) (panel (b)) when \( \sigma_{1} \) is artificially increased. From \( \sigma_{1} = 250 \text{ S/m} \) to \( \sigma_{1} = \sigma_{2}/100 \) there is only a weak modification of the growth rate and magnetic damping. At \( \sigma_{1} = \sigma_{2} \), the instability vanishes: a strong difference in conductivity is essential for the metal pad roll instability.

In Figure 6 we study how the rates \( \lambda_{v}/JB_{z} \) and \( \lambda_{vv}/B_{z}^{2} \) of the fundamental wave \((m, n) = (1, 1)\) vary with the electrical conductivity of the top (cryolite) layer. We investigate the range \( \sigma_{1} \in [250 \text{ S/m}, \sigma_{2}] \). We use the numerical approach to calculate \( \lambda_{vv}/B_{z}^{2} \), since the theoretical approximation (2.76) is not valid when \( \sigma_{1} \ll \sigma_{2} \). For \( \sigma_{1} \)
we compare with the theoretical estimates provided by the perturbative analysis. The viscous damping rate $\lambda$ is typically smeared over 2-3 cells. The interface with an aspect ratio of 5 at most. Interface compression ensures that the linear instability mechanism. In OpenFOAM only purely orthogonal cells are used with $h = 0.05$ or smaller. The cells are flattened towards the interface with an aspect ratio of 5 at most. Interface compression ensures that the interface is typically smeared over 2-3 cells.

Using both SFEMaNS and OpenFOAM, we simulate the decay of large scale gravity waves $(1, 1)$ with or without imposed magnetic field $B_z = 0, 10, 25, 50 \text{ mT}$ and using different grids with typical mesh-sizes $h = \delta x/R$. We measure the frequency $\omega + \delta_{\text{visc}}(s^{-1})$, the viscous damping rate $\lambda_{\text{visc}}(s^{-1})$ and the magnetic damping rates $\lambda_{\text{vv}}(s^{-1})$ of the waves and we compare with the theoretical estimates provided by the perturbative analysis.

<table>
<thead>
<tr>
<th>wave $(1, 1)$</th>
<th>$h$</th>
<th>$\omega + \delta_{\text{visc}}$</th>
<th>$\lambda_{\text{visc}}$</th>
<th>$\lambda_{\text{vv}}$ (10 mT)</th>
<th>$\lambda_{\text{vv}}$ (25 mT)</th>
<th>$\lambda_{\text{vv}}$ (50 mT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SFEMaNS</td>
<td>$(0.86 \rightarrow 5) \times 10^{-2}$</td>
<td>4.086</td>
<td>-0.0773</td>
<td>-0.1032</td>
<td>-0.1375</td>
<td>-0.2480</td>
</tr>
<tr>
<td></td>
<td>$(0.43 \rightarrow 5) \times 10^{-2}$</td>
<td>4.101</td>
<td>-0.0805</td>
<td>-0.0066</td>
<td>-0.0412</td>
<td>-0.1533</td>
</tr>
<tr>
<td></td>
<td>$(0.2 \rightarrow 5) \times 10^{-2}$</td>
<td>4.096</td>
<td>-0.0805</td>
<td>-0.0069</td>
<td>-0.0421</td>
<td>-0.1539</td>
</tr>
<tr>
<td>OpenFOAM</td>
<td>$5.00 \times 10^{-2}$</td>
<td>4.078</td>
<td>-0.0519</td>
<td>-0.0071</td>
<td>-0.0404</td>
<td>-0.1527</td>
</tr>
<tr>
<td></td>
<td>$4.00 \times 10^{-2}$</td>
<td>4.078</td>
<td>-0.0590</td>
<td>-0.0053</td>
<td>-0.0401</td>
<td>-0.1511</td>
</tr>
<tr>
<td></td>
<td>$3.33 \times 10^{-2}$</td>
<td>4.105</td>
<td>-0.0646</td>
<td>-0.0056</td>
<td>-0.0387</td>
<td>-0.1507</td>
</tr>
<tr>
<td></td>
<td>$2.50 \times 10^{-2}$</td>
<td>4.102</td>
<td>-0.0678</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$2.00 \times 10^{-2}$</td>
<td>4.107</td>
<td>-0.0717</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$1.539 \times 10^{-2}$</td>
<td>4.104</td>
<td>-0.0724</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Theory</td>
<td>4.101</td>
<td>-0.0682</td>
<td>-0.0066</td>
<td>-0.0414</td>
<td>-0.1658</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>wave $(2, 1)$</th>
<th>$h$</th>
<th>$\omega + \delta_{\text{visc}}$</th>
<th>$\lambda_{\text{visc}}$</th>
<th>$\lambda_{\text{vv}}$ (10 mT)</th>
<th>$\lambda_{\text{vv}}$ (25 mT)</th>
<th>$\lambda_{\text{vv}}$ (50 mT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SFEMaNS</td>
<td>$(0.86 \rightarrow 10.7) \times 10^{-2}$</td>
<td>5.087</td>
<td>-0.1981</td>
<td>-0.0039</td>
<td>-0.0331</td>
<td>-0.1290</td>
</tr>
<tr>
<td></td>
<td>$(0.43 \rightarrow 5) \times 10^{-2}$</td>
<td>5.150</td>
<td>-0.1251</td>
<td>-0.0084</td>
<td>-0.0478</td>
<td>-0.1807</td>
</tr>
<tr>
<td></td>
<td>$(0.2 \rightarrow 5) \times 10^{-2}$</td>
<td>5.084</td>
<td>-0.1236</td>
<td>-0.0085</td>
<td>-0.0473</td>
<td>-0.1804</td>
</tr>
<tr>
<td>Theory</td>
<td>5.267</td>
<td>-0.1056</td>
<td>-0.0079</td>
<td>-0.0492</td>
<td>-0.1968</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Using both SFEMaNS and OpenFOAM, we simulate the decay of large scale gravity waves $(m, n) = (1, 1), (2, 1)$ with or without imposed magnetic field $B_z = 0, 10, 25, 50 \text{ mT}$ and using different grids with typical mesh-sizes $h = \delta x/R$. We measure the frequency $\omega + \delta_{\text{visc}}(s^{-1})$, the viscous damping rate $\lambda_{\text{visc}}(s^{-1})$ and the magnetic damping rates $\lambda_{\text{vv}}(s^{-1})$ of the waves and we compare with the theoretical estimates provided by the perturbative analysis.

in the range $[250 \text{ S/m}, \sigma_2/100]$, we observe little changes in both $\lambda_v/|B_z|$ and $\lambda_{\text{vv}}/B_z^2$. This means that the conductivity of the electrolyte can be significantly increased without modifying the essence of the instability mechanism and the magnetic damping rate. But increasing further $\sigma_1$ does affect the instability; the instability disappears when $\sigma_1 = \sigma_2$, the metal pad roll instability cannot occur without a significant jump in conductivity.

4.1.2. *Numerical results & comparison with theory*

The numerical study is done with SFEMaNS and OpenFOAM ignoring surface tension. In both codes, we have varied the grid-size to assess the numerical convergence. We introduce the non-dimensional measure $h = \delta x/R$ to characterize the grid size. In SFEMaNS the cell-size $\delta x$ is non-uniform in the plane $(r, z)$ and the cells are refined near the interface. Therefore we provide intervals $h = (h_{\text{min}} \rightarrow h_{\text{max}})$ to characterize the SFEMaNS grid. All the simulations reported in the paper have been done with 30 real Fourier modes in the azimuthal direction. In SFEMaNS, we relax the conductivity jump by taking $\sigma_1 = \sigma_2/100$. This stabilizes the numerical calculations and as explained above, should not modify the linear instability mechanism. In OpenFOAM only purely orthogonal cells are used with $h = 0.05$ or smaller. The cells are flattened towards the interface with an aspect ratio of 5 at most. Interface compression ensures that the interface is typically smeared over 2-3 cells.
Figure 7. Visualization of the interface between the cryolite and the aluminum in the numerical simulations. (a) and (b) 8 snapshots over one period $T$ in the saturated regime for $J = -7795\,\text{A/m}^2$, $B_z = 16\,\text{mT}$. (c) Far from the stability threshold, the interface violently destabilizes and develops large deformations, $J = -20000\,\text{A/m}^2$, $B_z = 16\,\text{mT}$.
Metal pad roll instability in cylindrical reduction cells

Figure 8. (a) Comparison of theoretical and numerical growth rates for the large scale sloshing wave \((m,n) = (1,1)\) as a function of \(|J|\). (b) Saturation amplitudes of rotating wave as a function of \(|J|\). Small cell configuration with \(B_z = 16\, \text{mT}\) and ignoring surface tension.

In a first series of tests, we put \(J = 0, B_z = 0\), initialize the numerical simulations with low amplitude gravity wave profiles for \((m,n) = (1,1), (2,1)\) and let these waves decay in time. We measure numerical values for the frequency and viscous damping rates of these waves and compare these measures with the frequency \(\omega + \delta_{\text{visc}}\) and damping rate \(\lambda_{\text{visc}}\) of our theoretical model. All the results are gathered in Table 4. Both solvers OpenFOAM and SFEMaNS clearly show convergence on the gravity wave frequency \(\omega + \delta_{\text{visc}}\) as the grid is refined and the measured frequencies closely match with the theoretical value. For the fundamental wave \((1,1)\), the absolute error between the theory and the numerical values is of order \(0.01\, \text{s}^{-1}\) on the finest grids, and the relative errors is of the order \(2.4 \times 10^{-3}\). For the wave \((m,n) = (2,1)\) there is a larger mismatch but the agreement is still very good. For the viscous damping rates \(\lambda_{\text{visc}}\), we also observe convergence in both codes as the grid is refined. The waves are slightly more damped in the numerical simulations than predicted by the theory and there is a noticeable difference between both codes: the same waves are more viscously damped in SFEMaNS than in OpenFOAM.

In a second series of tests, we measure the damping rates of gravity waves in the presence of non-zero external magnetic fields \(B_z \neq 0\), but without electrolysis current \((J = 0)\). We perform simulations for increasing magnetic field \(B_z = 10, 25, 50\, \text{mT}\). Denoting by \(\lambda\) the (negative) growth rate of the waves in the numerical simulations using a mesh with cell-size \(h\) and \(\lambda_{\text{visc}}\) the viscous growth rate measured for this mesh, we then calculate the numerical values \(\lambda_{\text{vv}} = \lambda - \lambda_{\text{visc}}\) in order to filter the code- and grid-dependent viscous damping that was considered in the previous section. Table 4 gathers the numerical measures for \(\lambda_{\text{vv}}\). These measures compared with the theoretical estimates for waves \((1,1)\) and \((2,1)\). The agreement with the theoretical value is very good on sufficiently fine grids and excellent for the small values of the magnetic induction \(B_z = 10, 25\, \text{mT}\). For the largest value \(B_z = 50\, \text{mT}\) there is a small mismatch, which we suspect is due to the fact that we leave the asymptotic regime of validity of the theoretical model. We notice that the magnetic damping rates \(-\lambda_{\text{vv}}\) are slightly lower in OpenFOAM than in the simulations done with SFEMaNS.

In a third series of tests, we perform simulations of the metal pad roll instability with both \(J\) and \(B_z\) active. We fix the magnetic field strength to \(B_z = 16\, \text{mT}\) which is used in Steiner (2009). We increase the current density in the interval \(J \in [-7795, 25984] \, \text{Am}^{-2}\) which corresponds to electrolysis currents in the range \(I = [30, 100]\, \text{A}\). The current
runs downwards through the cell. From the theory, we estimate that the metal pad roll instability should occur when $|J|$ is above the threshold $J_c = 4476 \text{ Am}^{-2}$. Figure 7 shows snapshots from simulations at $|J| = 7795 \text{ Am}^{-2} > J_c$. The metal pad roll instability is clearly visible. We observe a saturated wave similar in shape to the fundamental wave $(m, n) = (1, 1)$, and this wave rotates in the anti-clockwise direction ($m\omega < 0$) when seen from above. This behaviour corresponds to what we expect from the theory and from the Sele-mechanism (Sele 1977).

In a more systematic series of computations we have measured the growth rate of the metal pad roll instability as a function of $J$. The results are shown in the panel (a) of Figure 8(a). We also show in this panel the theoretical estimate $\lambda \approx \lambda_v + \lambda_{visc}$. We observe an excellent agreement with the theoretical line which, we recall, grows linearly in $J$ because $\lambda_v \sim |J B_z|$. The fact that both codes find straight lines with the same slope is an indication that both codes capture the destabilizing mechanism at the origin of $\lambda_v$ very well. The growth rates from SFEMaNS are slightly beneath those from OpenFOAM, which is not unexpected owing to the small difference in viscous damping rates we have observed before.

The numerical simulations further provide insights in the nonlinear regime, which is not modelled in our theory. In Figure 8(b), we show a diagram that gathers the maximal amplitude $\eta_{\text{max}}$ of the saturated rotating gravity wave as a function of current density. This curve suggests a supercritical bifurcation. We propose a weakly nonlinear fit $\eta_{\text{max}} = 1.5 \times 10^{-4} \sqrt{|J| - J_c}$ in MKS units (based on the OpenFOAM data). In non-dimensional form and using the theoretical formula for the growth rate $\lambda$, we rewrite this fit as follows:

$$\text{If } \lambda > 0 : \quad \frac{\eta_{\text{max}}}{R} \approx 2 \sqrt{\frac{\lambda}{|\omega|}}.$$  \hfill (4.6)

It is possible that this non-dimensional relation could be more generally applicable to estimate the weakly nonlinear saturation amplitudes of the fundamental wave $(1, 1)$ in tall (deep) reduction cells.

The weakly nonlinear regime exists only in a small parameter range. When we push the electrical current density even higher, say above $|J| \sim 15000 \text{ Am}^{-2}$, we can get violent secondary instabilities with both numerical solvers. The interface undergoes strong deformations which ultimately lead to a blow up of the numerical calculation. To illustrate this phenomenon, we show in Figure 7(c) the interface obtained with $J = -20000 \text{ Am}^{-2}$ using SFEMaNS. The origin of this secondary instability is unknown, but similar explosive behaviour was also observed in liquid metal battery geometry Weber et al. (2017a,b).

4.2. Large, industrial scale cells

We now apply the theory to large, industrial scale cells, with "realistic" dimensions

$$R \in [0.5, 2.5] \text{ m}, \quad H_1 = 0.05 \text{ m}, \quad H_2 = 0.30 \text{ m}.$$  \hfill (4.7)

Since the cell is very large, we can ignore surface tension. All the material parameter are the same as in (4.2).

In Table 5 we give the numerical values for all the relevant quantities determining the stability and frequency shifts for different waves $(m, n)$ in a cell of radius $R = 1 \text{ m}$. In the column for $\lambda_v/|J B_z|$, we see that the fundamental wave $(1, 1)$ is always the most unstable. Since the viscous dissipation is also the lowest for this wave, we expect to see this mode to be dominant in an unstable cell. The relative frequency shift $\delta_h/J^2$ can be either positive or negative and it decays as $m$ or $n$ increase. Although the numbers are
small, this effect is no longer negligible for current densities of the order $J \sim 10^4$ Am$^{-2}$. For the quasi-static corrections $\lambda_{vv}/B_z^2, \lambda_{hh}/J^2$, and $\delta_{vh}/JB_z$ we find the same trends as in the small cell. With $\lambda_{hh}/J^2 = 8.06 \times 10^{-14}$ s$^{-1}$ m$^2$ A$^{-2}$ we also find here that the wave (1, 1) may become unstable for large values of the current even with $B_z = 0$. However, this would also require currents so large that one would have $\delta_{h} \gg \omega$, which would then make the perturbation method no longer trustworthy.

We fix $J = 1.5 \times 10^4$ Am$^{-2}$ and $B_z = 1$ mT, which we consider to be realistic values, and we let the radius of the cell vary in the range $R \in [0.5, 2.5]$ m. Due to the low strength of the vertical magnetic field and the low values of $\lambda_{hh}/J^2$, we can neglect all the quasi-static MHD effects in the large cell and use instead the following approximation:

$$\lambda \approx \lambda_v + \lambda_{visc}, \quad \delta \approx \delta_h + \delta_{visc}. \quad (4.8)$$

Figure 9 shows how the growth rate $\lambda$ and the shifted frequency $\omega + \delta$ vary with $R$. In panel (a), we clearly see that the growth rate increases almost linearly with $R$ in agreement with the shallow limit formula (2.55). The large scale sloshing mode (1, 1) is again the most unstable mode, even for the large cells under consideration. In panel (b), the dashed
Table 6. Comparison of exact theoretical formula with shallow limits for inviscid frequencies $\omega$ (in s$^{-1}$) and relative growth rates $\lambda_v/JB_z$ (in s$^{-1}$A$^{-1}$m$^2$T$^{-1}$), relative magnetic damping term $\lambda_{vv}/B_z^2$ (in s$^{-1}$T$^{-2}$), viscous damping term $\lambda_{\text{visc}}$ (in s$^{-1}$) and relative frequency shift $\delta_h/J^2$ (in s$^{-1}$A$^{-2}$m$^4$). In the large cell under study, the shallow limit formula provide a very accurate description.

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>exact</th>
<th>$\lambda_v/JB_z$</th>
<th>$\lambda_{vv}/B_z^2$</th>
<th>$\lambda_{\text{visc}}$</th>
<th>$\delta_h/J^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td></td>
<td>0.311 $1.79 \times 10^{-3}$</td>
<td>$-18.83$</td>
<td>$-0.767 \times 10^{-2}$</td>
<td>$2.27 \times 10^{-10}$</td>
</tr>
<tr>
<td>shallow</td>
<td></td>
<td>0.314 $1.75 \times 10^{-3}$</td>
<td>$-18.79$</td>
<td>$-0.751 \times 10^{-2}$</td>
<td>$2.30 \times 10^{-10}$</td>
</tr>
<tr>
<td>(2, 1)</td>
<td></td>
<td>0.508 $1.01 \times 10^{-3}$</td>
<td>$-28.78$</td>
<td>$-0.983 \times 10^{-2}$</td>
<td>$0.58 \times 10^{-10}$</td>
</tr>
<tr>
<td>shallow</td>
<td></td>
<td>0.521 $0.95 \times 10^{-3}$</td>
<td>$-28.70$</td>
<td>$-0.967 \times 10^{-2}$</td>
<td>$0.59 \times 10^{-10}$</td>
</tr>
</tbody>
</table>

Figure 10. Large cell geometry $R = 1$ m. Streamlines of the current perturbations together with normalized power density plots for the fundamental wave $(1, 1)$.

Lines show that the inviscid frequency $\omega$ decays with $R$. This is coherent with theory that predicts $\omega_{\text{shallow}} \sim R^{-1}$. The solid lines show the shifted frequency, $\omega + \delta$, which clearly deviates from $\omega$. The shift is here mostly due to the contribution $\delta_h$, i.e., it is caused by the Lorentz-force interaction of the electrical currents $\hat{j}_i$ with the horizontal magnetic field $\mu_0 r/2 e_\theta$. Notice finally, that the inequalities $\lambda \ll \omega$ and $\delta \ll \omega$ are verified in the entire range of radii considered; that is, the perturbation method is applicable to model realistically large reduction cells.

In Figure 10, we show the current perturbations $\hat{j}_i$, $\hat{J}_i^v$, $\hat{J}_i^h$ together with the power densities (4.4) as we did in Figure 5 for the small cell configuration. In panel (a), we see that $\hat{j}_i$ is mainly vertical in the electrolyte and horizontal in the aluminium. This is remarkably different from what we had in the small cell, but also corresponds to what is always assumed in all the shallow layer models of the metal pad roll instability available in the literature. Even though $\hat{J}_i^v$ and $\hat{J}_i^h$ have negligible impact, it is reassuring to see that these currents are also almost horizontal and remain confined to the aluminium layer. In panel (b), we plot the power densities (4.4), i.e., the integrands defining $P_v$, $Q_{vv}$ and $Q_{hh}$. Contrary to the small cell configuration, we see that the main sources of instability are concentrated in $P_v$ and are mainly localized in the bottom layer where the currents $\hat{j}_i$ are horizontal.
4.3. (Mg-Sb) liquid metal battery

The references Weber et al. (2017a, b) numerically investigate the metal pad roll instability in a Mg||NaCl – KCl – MgCl₂||Sb liquid metal battery (LMB) of cylindrical shape. These batteries are structurally similar to reduction cells, except that they are composed of three layers of conducting fluids (top, middle, bottom = indices 1, 2, 3) instead of two. The cell studied in the above references has dimensions

\[ R = 0.05 \text{ m}, \quad (H_1, H_2, H_3) = (0.045, 0.01, 0.045) \text{ m}, \]

The material properties are

\[ (\rho_1, \rho_2, \rho_3) = (1577, 1715, 6270) \text{ kg m}^{-3}, \]
\[ (\sigma_1, \sigma_2, \sigma_3) = (3.6 \times 10^6, 80, 8.7 \times 10^5) \text{ Sm}^{-1}, \]
\[ (\nu_1, \nu_2, \nu_3) = (6.7, 6.8, 2.0) \times 10^{-7} \text{ m}^2 \text{s}^{-1}, \]

with the surface tensions

\[ (\gamma_{1|2}, \gamma_{2|3}) = (0.19, 0.095) \text{ N m}^{-1}. \]

The top (Mg) and bottom (Sb) layers are very good conductors, but the middle layer, composed of molten salt (NaCl – KCl – MgCl₂), is a badly conducting electrolyte. The numerical simulations performed in Weber et al. (2017a) are done with \( B_z = 10 \text{ mT} \) and varying electrical currents \( I = J \pi R^2 \). The metal pad roll instability is observed and growth rates are reported therein. Due to the large density ratio between the second and third layer, the instability mainly deforms the upper interface, separating the first and the second layer. This phenomenon is visible in Figure 11-(a), showing a snapshot of a saturated rotating wave. The lower interface only slightly deforms and the whole bottom layer doesn’t really participate to the dynamics. Actually, closer inspection reveals that the frequency of the wave matches very closely that of the two layer system composed of the first and the second layer. This observation has motivated us to try to apply our two layer reduction cell theory to this three-layer LMB.

The following "technical" operations are necessary in order to apply the theory. We entirely ignore region 3 and all what may occur there. At the interface of regions 2 and 3, the electrical boundary condition, as seen from the electrolyte is very similar to the one we had on the top cap of our cylindrical Hall-Heroult cell: the electrolyte meets there a much better conductor which we henceforth assume to be perfect. This means that if we exchange regions 1 and 2 and inverse the sense of gravity i.e., \( g \rightarrow -g \), the theory and the formula developed in the previous sections apply. The only difference is that the sense of rotation of the unstable wave has to be inversed. Finally, all the horizontal field effects are negligible here, just as in the small reduction cell.

In Figure 11 we compare the theoretical growth rates with simulations done with OpenFOAM for various values of the current. The dashed line corresponds to our theoretical estimate \( \lambda = \lambda_v + \lambda_{vv} + \lambda_{visc} \). The data-points are those from Figure 9 in Weber et al. (2017a). The agreement is surprisingly good. The fact that we find the right slope with our two-layer theory implies that \( \lambda_v \) is correctly calculated. The threshold also seems approximatively right, which suggests that the combined magnetic and viscous damping rates \( \lambda_{vv} + \lambda_{visc} \) are also correct. Since the magnetic damping associated with \( \lambda_{vv} \) is only active in the well-conducting Mg layer, where the fluid is stirred, it seems unlikely that this term could not be estimated correctly, which then implies that \( \lambda_{visc} \) should also be nearly correct. This is more surprising because the viscous damping rate formula used here, supposes that the no-slip condition applies both on the top and the bottom walls. Here in this LMB, the correct viscous boundary condition at the interface
between regions 2 and 3 is not the no-slip condition, but should rather be the expression of the continuity of the velocity and the normal stress. Now, considering that the bottom fluid 3 is significantly heavier and has a much larger dynamical viscosity $\rho_3 \nu_3$ than the fluid occupying region 2, it is legitimate to expect that the third layer is only very weakly entrained by the motion of the salt. The continuity of the velocity then suggests that the no-slip boundary condition is indeed a good approximation of the reality. Therefore, we think that the viscous damping is also more or less correctly calculated here.

5. Conclusion

In this article, we have formulated a theoretical model for the metal pad roll instability in cylindrical reduction cells. Using perturbation methods, we have derived analytical formulas for the destabilization, the magnetic damping and the viscous damping of gravity waves in cylinders. Our model is new. This model is complete in the sense that it overcomes all the common limitations of the metal pad roll theories found in the literature: no assumptions are made on the shallowness of the layers, the capillary effects are included, the viscous effect are accounted for properly, the theory goes beyond the commonly adopted magneto-static approach and works for arbitrary fluids. We have tested the theoretical formulas against numerical calculations of the interaction integrals. From the general formulas, we have derived asymptotic expressions for the deep and the shallow limits. In the Appendix B, we demonstrate that our model degenerates correctly to the shallow and inviscid metal pad roll theories in cylindrical geometry that are now well established in the literature.

The theoretical expression for the viscous damping rates have been tested in experiments, using an orbital sloshing device fully described in Horstmann et al. (2018b). We have found excellent agreements with our theoretical formulas. Improvements are still possible by taking into account damping caused by viscous dissipation near the free interface (doable) and by finding a better boundary layer model near the moving contact line (difficult).

We have applied our theoretical model to a small reduction cell, representative of laboratory scale experiments. This cell was previously studied numerically by Flueck et al. (2009) and Steiner (2009). Being very small, this cell requires large magnetic fields
to be destabilized, just like those used in the experiments reported in Pedchenko et al. (2009, 2016). One particularity of the small cell is that the fundamental wave $(1, 1)$ is not the only wave to become unstable nor is it the most unstable wave. We have also shown that capillary effects have a very significant impact in small cells. The small reduction cell is a good benchmark for our solvers SFEMaNS and OpenFOAM. Both codes yield sensibly the same results, and in a detailed cross-comparisons we have found excellent agreements with our theoretical model. This combined theoretical/numerical approach has been very valuable since it allowed us to gradually perfect and debug our numerical solvers and our theory.

For large industrial scale cells we have limited ourselves to some theoretical aspects only. A first important point we have made is that even large reductions cells may be modelled by using the perturbation method in limited parameter regimes. Viscous effects and quasi-static corrections always remain small in large cells in agreement with what is frequently assumed. Our theory predicts that the large scale sloshing mode $(1, 1)$ is always by far the most unstable wave in large cells. The self-generated horizontal magnetic field $\mu_0 J r/2 e_\theta$ cannot destabilize waves at the first order, but it modifies the wave frequencies through the additional term $\delta_h$. This has not been tested numerically, but is in agreement with previous findings (Sneyd 1985; Sneyd & Wang 1994). We expect that this frequency shift is indeed measurable in large cells.

A remarkable detail is that we have found that quasi-static effects may destabilize the fundamental wave $(1, 1)$ very weakly even in the absence of $B_z$. Numerical investigations suggest that this unreported phenomenon is negligible. Probably this phenomenon may be related in some weak way to the Tayler instability investigated in Weber et al. (2013, 2014); Herreman et al. (2015). The Tayler instability has also an inductive origin (quasi-static MHD is necessary), it is also caused by the term $J \times b$, and it is also only destabilizing for $m = 1$ modes.

Finally we have also applied our theoretical model to the metal pad roll instabilities that can occur in Mg–Sb liquid metal batteries. There, the bottom layer is so heavy that it does not participate actively in the destabilization and stabilization of the gravity waves. With some well justified minor technical modifications we have found that our model explains very well the growth rates numerically obtained in Weber et al. (2017a).

Our study opens a few perspectives. First of all, we think that our precise theory can be used to benchmark multiphase MHD codes. Second, it is possible to apply the proposed perturbation method to cells with a rectangular geometry in order to derive a theoretical model for the experiments reported in Pedchenko et al. (2009, 2016). Or alternatively, perhaps these experiments could be done in a cylindrical geometry to compare with our model. Third, we plan to extend our two-layer model to three layers of fluid to be able to model the metal pad roll instability in arbitrary liquid metal batteries. As shown by several authors (Horstmann et al. 2018a; Molokov 2018; Zikanov 2018), the waves on the two interfaces may interact more or less depending on the density differences between the three layers. To capture correctly the metal pad roll instability in arbitrary liquid metal batteries one must include this possibility.

Acknowledgments

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Appendix A. Details on the analytical calculation of $\lambda_v$ and $\delta_h$

We provide here some technical details on the computation of $\mathcal{P}_v, \mathcal{P}_h, \mathcal{K}$, which we recall give the growth rate $\lambda_v = \frac{P_v}{2K}$ and the frequency shift $\delta_h = \frac{P_h}{2K}$ in the quasistatic inviscid limit. With the definition (2.42) for $\mathcal{K}$ and the expression for $\hat{\eta}$, we find

$$\mathcal{K} = \left[(\rho_2 - \rho_1)g + \gamma_1|2k^2\right] \frac{2\pi |A|^2 k^2}{\omega^2} \int_0^R J_m^2(kr) r \, dr. \quad (A.1)$$

The radial integral can be evaluated analytically. Using Equation (11) from Watson (1995), p. 135, together with the identity $J'_m(kR) = 0$, we obtain

$$\int_0^R J_m^2(kr) r \, dr = \frac{1}{2k^2} (k^2 R^2 - m^2) J_m^2(kR). \quad (A.2)$$

This formula also demonstrates that $(k^2 R^2 - m^2) > 0$, which is a property that we have used repeatedly in the paper. The two volume integrals in the expression (2.44a) that defines $\mathcal{P}_v$, are transformed into surface integrals by remarking that the two integrands can be written in conserved form: $\nabla \hat{\phi} \pi \cdot (\sigma_i \nabla \hat{\varphi}_i \times B_z e_z) = \nabla \cdot [\sigma_i \hat{\phi} \nabla \hat{\varphi}_i \times B_z e_z]$. This yields

$$\mathcal{P}_v = 2\pi(-im) B_z \left( \sigma_1 \int_0^{H_1} \hat{\phi}_1 \hat{\varphi}_1 |_{r=R} \, dz + \sigma_2 \int_{-H_2}^0 \hat{\phi}_2 \hat{\varphi}_2 |_{r=R} \, dz \right)$$

$$= \frac{2\pi m J B_z |A|^2 A J_m^2(kR)}{2\omega} \left[ \tanh(kH_1) + \frac{(kH_2)}{\sinh^2(kH_2)} + \frac{1}{\tanh(kH_2)} \right] \quad (A.3)$$

with $A$ defined in (2.49a). The calculation of $\mathcal{P}_h$, which is needed to compute the frequency shift $\delta_h$, is more involved since there are more terms to evaluate. The following identity:

$$\int_0^R J_m(kr) \partial_r [J_m(kr)] r^2 \, dr = \frac{m^2}{2k^2} J_m^2(kR), \quad (A.4)$$

is useful to simplify one radial integral that is encountered on the way. The rest of the computation is omitted for brevity. We have verified the correctness of the expressions (2.49) for $\lambda_v$ and $\delta_h$ by performing numerical approximations of the integrals $\mathcal{P}_v, \mathcal{P}_h, \mathcal{K}$ on a fine discrete grid.

Appendix B. Linking $\lambda_v$ to existing shallow models

Both Davidson & Lindsay (1998) and Lukyanov et al. (2001) provide shallow models for the metal pad roll instability in cylindrical geometry in the inviscid and magnetostatic limit and without surface tension. Here we show that our expression of the growth rate (2.49) coincides with the above models when passing to the limit.

We start from the model of Lukyanov et al. (2001) and to avoid conflicts of notation, we adapt the formulas therein to our setting. The two-dimensional problem to be solved in the shallow approximation is

$$\partial_{tt} \eta - c^2 \nabla^2 \eta = 0, \quad \nabla^2 \chi = \beta \eta. \quad (B.1)$$
Here $\eta(r, \theta, t)$ is the perturbation of the surface elevation, and $\chi(r, \theta, t)$ is the perturbation on the electrical potential. The wave-speed $c$ and the parameter $\beta$ are defined by

$$c^2 = \frac{(\rho_2 - \rho_1)g}{\rho_1 H_1^{-1} + \rho_1 H_2^{-1}}, \quad \beta = \frac{JB_z}{(\rho_2 - \rho_1)g H_1 H_2}. \quad (B.2)$$

Note that the quantity $\beta$ used here is not a non-dimensional number. On the right-hand side of the equation for $\chi$ in (B.1), we find $+\beta \eta$ (instead of $-\beta \eta$ in Lukyanov et al. (2001)), because in the present article we use the convention that the electrolysis current density is $J = J_e z$ (instead of $J = -J_e z$ in Lukyanov et al. (2001)). On the lateral sidewall $r = R$, the following conditions must be met:

$$\partial_r \chi|_{r=R} = 0, \quad \partial_r \eta|_{r=R} = -R^{-1} \partial_{\theta} \chi|_{r=R}. \quad (B.3)$$

The solutions to (B.1) take the form

$$\eta = AJ_m(kr) e^{i(m\theta + \omega t)}, \quad \chi = (-A^{-1} \beta^2 J_m(kr) + C r^m) e^{i(m\theta + \omega t)}, \quad (B.4)$$

with $k^2 = \omega^2/c^2$ and $A$, $C$ are arbitrary constants. Note that for the exponential factors we have chosen the convention $\exp(i(m \theta + \omega t))$ (instead of $\exp(i(m \theta - \omega t))$ in Lukyanov et al. (2001)). By enforcing the boundary conditions, we find the non-dimensional dispersion relation:

$$\beta J_{m+1}(\kappa) = -i\kappa^2 J_m'(\kappa), \quad (B.5)$$

with $\kappa = kR$ and

$$\beta = \frac{JB_z}{(\rho_2 - \rho_1)g H_1 H_2}. \quad (B.6)$$

Notice that the multiplicative factor $\beta$ is the same as that appearing in the shallow limit of $\lambda_v$ with zero surface tension, $\gamma_{1|2} = 0$, (see (2.56)). We now derive an approximate solution of (B.5) in the limit $\beta \ll 1$. We start with the expansion

$$\kappa = \kappa^{(0)} + \beta \kappa^{(1)} + O(\beta^2), \quad (B.7)$$

which we insert into (B.5). Then we expand the dispersion relation using Taylor series in powers of $\beta$. At the order $O(\beta^0)$, we find the constraint

$$0 = -i \left( \kappa^{(0)} \right)^2 J_m'(\kappa^{(0)}) \implies \kappa^{(0)} = \kappa_{mn}. \quad (B.8)$$

Hence, for any $m$ the leading order of the non-dimensional radial wavenumber is the same as the one we have obtained in our model (see (2.13)). At order the $O(\beta^1)$, we have the constraint

$$J_{m+1}(\kappa_{mn}) = -i \left[ \kappa^{(1)} \kappa_{mn}^2 J_m'(\kappa_{mn}) + 2 \kappa^{(1)} \kappa_{mn} J_m'(\kappa_{mn}) \right]. \quad (B.9)$$

Using $J_m'(\kappa_{mn}) = 0$ together with the recurrence relations and the Bessel differential equation, we have

$$J_{m+1}(\kappa_{mn}) = \frac{m}{\kappa_{mn}} J_m(\kappa_{mn}), \quad J_m'(\kappa_{mn}) = -\left( \frac{\kappa_{mn}^2 - m^2}{\kappa_{mn}^2} \right) J_m(\kappa_{mn}). \quad (B.10)$$

This in turn yields the following simplified expression for $\kappa^{(1)}$:

$$i\kappa^{(1)} = \frac{m}{\kappa_{mn}(\kappa_{mn} - m^2)}. \quad (B.11)$$

The small shift in wavenumber $\kappa^{(1)}$ is equivalent to a small shift in the complex frequency
\( \omega \) of the waves, since by definition \( \omega = \pm \frac{c \kappa}{R} \) in the context of the shallow layer approximation. Using the expansion

\[
\omega = \omega^{(0)} + \beta \omega^{(1)} + O(\beta^2),
\]

we obtain

\[
\omega^{(0)} = \omega_{\text{shallow}} = \pm \frac{c \kappa_{mn}}{R}, \quad \omega^{(1)} = \pm \beta \frac{c \kappa_{(1)}^{(1)}}{R}.
\]

We recognize \( \omega^{(0)} \) as the eigenfrequency \( \omega_{\text{shallow}} \) of the non-dispersive gravity waves in the shallow layer limit. The complex frequency shift \( \alpha \) is defined in our model to be

\[
\alpha = i \omega - i \omega^{(0)} = \beta \frac{m \omega^{(0)} \kappa_{2}^{2}}{\kappa_{mn}^{2} - m_{2}^{2} \kappa_{mn}^{2}}.
\]

This is exactly the shallow limit of the growth rate \( \lambda_v \) with \( \gamma_{1|2} = 0 \) as stated in (2.53a); recall that \( \kappa_{mn} := kR \). In conclusion, our results coincide with those of Lukyanov et al. (2001); Davidson & Lindsay (1998) in the inviscid, magnetostatic, shallow, zero surface tension, and small \( \beta \) approximation.

### Appendix C. Other electrical conditions on top and bottom caps

The top and bottom boundary conditions (2.6c) and (2.6d) for the electrical potential and currents (2.6c) are well adapted to model the metal pad roll instability in Hall-Heroult cells, because they mimic the fact that the top Carbon anodes are better conductors than the cryolite, and the bottom cathode is not as good a conductor as the aluminium. Nevertheless, this is an approximation and we propose in this section to study the impact of using different top and bottom boundary conditions. More precisely we consider the following two sets of boundary conditions:

\[
\begin{align*}
\text{(zero } \phi) & \quad \phi_1|z=H_1 = 0, \quad \phi_2|z=-H_2 = 0, \\
\text{(zero } j_z) & \quad j_{1,z}|z=H_1 = 0, \quad j_{2,z}|z=-H_2 = 0.
\end{align*}
\]

For these boundary conditions, we find the potentials in the magneto-static inviscid limit to be

\[
\begin{align*}
\text{(zero } \phi) & \quad \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \frac{JkA}{\omega} \begin{bmatrix} \sigma_1^{-1} - \sigma_2^{-1} \\ \sigma_1^{-1} \tanh(kH_1) + \sigma_2^{-1} \tanh(kH_2) \end{bmatrix} \\
& \quad \times \begin{bmatrix} -i \sinh(k(z - H_1)) / (\sigma_1 \cosh(kH_1)) \\ -i \sinh(k(z + H_2)) / (\sigma_2 \cosh(kH_2)) \end{bmatrix} J_m(kr)e^{im\theta}, \\
\text{(zero } j_z) & \quad \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \frac{JkA}{\omega} \begin{bmatrix} \sigma_1^{-1} - \sigma_2^{-1} \\ \sigma_1^{-1} \tanh^{-1}(kH_1) + \sigma_2^{-1} \tanh^{-1}(kH_2) \end{bmatrix} \\
& \quad \times \begin{bmatrix} i \cosh(k(z - H_1)) / (\sigma_1 \sinh(kH_1)) \\ -i \cosh(k(z + H_2)) / (\sigma_2 \sinh(kH_2)) \end{bmatrix} J_m(kr)e^{im\theta}.
\end{align*}
\]

We then recalculate \( P_v \) and find the growth rate

\[
\lambda_v = \frac{\omega}{2} \left( \frac{JB_z}{\rho_2 - \rho_1} g + \gamma_{1|2} k^2 \right) \frac{m}{(kR)^2 - m^2} \Xi, \]

\( \Xi \)
with

\[
\Xi = \frac{(\sigma_1^{-1} - \sigma_2^{-1}) \left[ \sum_{i=1,2} \tanh(kH_i) \right]}{\sum_{i=1,2} \sigma_i^{-1} \tanh(kH_i)}, \quad (C.4a)
\]

\[
\Xi = \frac{(\sigma_1^{-1} - \sigma_2^{-1}) \left[ \sum_{i=1,2} (kH_i) \sinh^{-2}(kH_i) + \tanh^{-1}(kH_i) \right]}{\sum_{i=1,2} \sigma_i^{-1} \tanh^{-1}(kH_i)}. \quad (C.4b)
\]

In the deep cell limit, \(kH_i \gg 1\), and for large conductivity jumps, \(\sigma_1 \ll \sigma_2\), we find that

\[
\lambda_v,\text{deep} \approx \frac{J B_z}{(\rho_2 - \rho_1)g + \gamma_{1/2}k^2 (kR)^2 - m^2} \left[ \frac{1}{2} \left( 1 + \frac{H_2}{H_1} \right) \right] \frac{m \omega_{\text{shallow}}}{\kappa_{nn} - m^2}, \quad (C.5)
\]

for both sets of boundary conditions. Notice that this expression is exactly the same formula as (2.51a). The fact that the top and the bottom boundary conditions do not have an impact on the growth rate in deep cells is not a surprise: all the motion occurs near the interface, far away from the top and bottom boundaries. In the shallow cell limit, \(kH_i \ll 1\), and for large conductivity jumps, \(\sigma_1 \ll \sigma_2\), we have

\[
\lambda_v,\text{shallow} \approx \frac{J B_z}{(\rho_2 - \rho_1)g + \gamma_{1/2}k^2 (1 + H_2/H_1)} \left[ 1 + \frac{H_1}{H_2} \right] \frac{m \omega_{\text{shallow}}}{\kappa_{nn}^2 - m^2}, \quad (C.6a)
\]

\[
\lambda_v,\text{shallow} \approx \frac{J B_z}{(\rho_2 - \rho_1)g + \gamma_{1/2}k^2} \left[ \frac{1}{2} \left( 1 + \frac{H_2}{H_1} \right) \right] \frac{m \omega_{\text{shallow}}}{\kappa_{nn}^2 - m^2}. \quad (C.6b)
\]

These formulas are very different from the shallow cell formula we have computed with mixed boundary conditions, (see (2.56)). Comparing these three expressions, we conclude that the mixed boundary conditions (2.6c)-(2.6d) always yield the most unstable configuration in the shallow cell limit.

Appendix D. Calculation of viscous damping and frequency detuning

To find the leading order of the viscous damping, we must account for dissipation in the boundary layer regions which come in four different types: (i) There are strong Stokes boundary layers near the rigid walls \(\Sigma_i\); (2) There are weak Stokes boundary layers near the interface \(S\); (3) There are complex boundary layers in the corners of the rigid surface \(\Sigma_i\); (4) There are very complex (moving) boundary layers near the meniscus \(M\). Damping occurring in the regions 1 and 2 is straightforward to model, but the corners and meniscus regions 3 and 4 are very difficult to deal with, and so they are almost never considered in theoretical calculations.

Using the energy method of Lamb (1945), Case & Parkinson (1957) have calculated the viscous damping of gravity waves in cylinders. This calculation includes dissipation caused by boundary layers in the regions 1 (strong influence) and 2 (weak influence), but ignores what happens in the regions 3 and 4. The purpose of this section is twofold: first we extend the above approach to our two-fluid system, then we propose an alternative approach that better fits our perturbative framework. Our approach illustrates well why it is dangerous to ignore the complex boundary layer structure near the contact line in region 4.

D.1. First approach: viscosity modifies the solvability condition

To simplify the notation we temporarily omit the use of the indices \(i\) since both fluid regions are dealt with similarly. We focus on the effect of the boundary layers in the
region 1, near the rigid walls $\Sigma_1$. We modify the asymptotic ansatz \((2.58)\) for the velocity and pressure variables as follows:

$$[u + \bar{u}, p + \bar{p}, \ldots] = \left(\left[\tilde{u} + \hat{u}, \tilde{p} + \hat{p}, \ldots\right] + \left[\tilde{u} + \bar{u}, \tilde{p} + \bar{p}, \ldots\right]\right)e^{i(\omega + \alpha)t}. \quad (D.1)$$

The fields $\hat{u}, \hat{p}$ are again the inviscid gravity wave profiles. Barred variables refer to boundary layer corrections that only exist in $O(\sqrt{\nu/\omega})$ wide regions in the immediate vicinity of the rigid walls and the free surface. These corrections are required in order to meet the no-slip condition

$$u\big|_\Sigma + \bar{u}\big|_\Sigma = 0. \quad (D.2)$$

We start by calculating the leading order structure of $u\big|_\Sigma$. Let $x$ be the position vector in the fluid regions, $x_\Sigma$ be the position vector of the rigid surface $\Sigma$, and $n$ be the unit outward normal. We introduce the rescaled wall-normal coordinate

$$\zeta = \sqrt{\frac{\nu}{\omega}} (x_\Sigma - x) \cdot n. \quad (D.3)$$

When the viscosity is small or, more precisely, when the Reynolds number

$$Re = \frac{|\omega| R^2}{\nu} \gg 1, \quad (D.4)$$

is large, the wall normal coordinate $\zeta$ increases rapidly away from the boundaries. As such, this rescaled variable is well adapted to capture rapid variations in the boundary layers. We suppose that the dependence of the boundary layer corrections with respect to the normal direction is well represented by the rescaled $\zeta$, (this is only true away from the corners and the meniscus). To express the momentum and mass balance near the wall regions, we use the following approximations:

$$\nabla \cdot \bar{u} \approx \nabla \cdot \tilde{u} - \sqrt{\frac{\nu}{\omega}} \varphi_\zeta \bar{n}, \quad \nabla \tilde{p} \approx \nabla \cdot \tilde{p} - n \sqrt{\frac{\nu}{\omega}} \partial_\zeta \bar{p}, \quad \nabla^2 \bar{u} \approx \sqrt{\frac{\nu}{\omega}} \partial^2_\zeta \bar{u}. \quad (D.5)$$

Here $\perp$ refers to the part of the vector field that is tangential to the boundary and $\bar{n} = \bar{u} \cdot n$. Then using the fact that the tilded fields are $O(Re^{-1/2})$ smaller than the hatted fields, the leading orders of the momentum and mass balance are

$$\partial_t \hat{p} = 0, \quad \partial_\zeta (\hat{u} \cdot n) = 0, \quad i \text{Sgn}(\omega) \hat{u}_\perp = \partial^2_\zeta \hat{u}. \quad (D.6)$$

Using the no-slip condition on the rigid surface, $u\big|_\Sigma + \tilde{u}\big|_{\zeta=0} = 0$, together with the constraint that the boundary layer corrections decay to zero as $\zeta \to +\infty$, we find the leading order of the boundary layer corrections to be

$$\hat{p} = 0, \quad \hat{u} \cdot n = 0, \quad \hat{u}_\perp = -\tilde{u}_\perp \exp(-\Gamma \zeta), \quad (D.7)$$

with $\Gamma = (1 + i \text{Sgn}(\omega))/\sqrt{2}$. This correction produces a small flow normal to the boundary $\tilde{u} \cdot n$ that can be calculated using mass-conservation. Incompressibility implies that

$$\partial_\zeta (\tilde{u} \cdot n) = \frac{\nu}{|\omega|} \nabla \cdot \tilde{u}_\perp \quad \Rightarrow \quad \tilde{u} \cdot n = \frac{1}{\Gamma} \sqrt{\frac{\nu}{|\omega|}} \nabla \cdot \tilde{u}_\perp \exp(-\Gamma \zeta). \quad (D.8)$$

Expressing impenetrability $\tilde{u} \cdot n\big|_\Sigma + \tilde{u} \cdot n\big|_{\zeta=0} = 0$, we then find that

$$\tilde{u}_i \cdot n_i\big|_{\Sigma_i} = -\frac{1}{\Gamma} \sqrt{\frac{\nu_i}{|\omega|}} \nabla \cdot \tilde{u}_i \perp\big|_{\Sigma_i}, \quad (D.9)$$

where the indices $i = 1, 2$ refer to the different fluid regions. This relation has the following meaning: a small inviscid flow (with tildes) is forced in the bulk (without bars) as a
reaction to a small mass flux of viscous origin that seeps out from the boundary layers. This phenomenon is usually called “boundary layer pumping”.

The viscous modification of the boundary condition enters our perturbative model on the metal pad roll instability through the solvability condition \( (2.64) \) by means of the boundary terms marked by \( T_2 \). We have to replace \( T_2 = \alpha K \) in \( (2.42) \) by

\[
T_2 = \alpha K + D,
\]

where

\[
D = \sum_{i=1,2} \int_{\Sigma_i} \hat{\rho}_i \hat{u}_i \cdot n_i |_{\Sigma_i} dS = -\frac{1}{T} \sum_{i=1,2} \sqrt{\nu_i / |\omega|} \int_{\Sigma_i} \hat{\rho}_i \nabla \cdot \hat{u}_i |_{\Sigma_i} dS \tag{D.11}
\]
is the power dissipated in the boundary regions near the rigid walls. Let us now express the contributions of the different parts of the boundary and in both fluids. Considering that the flow is potential, so that \( \hat{u}_1 = \nabla \cdot \hat{\phi}_1 \), and the fact that \( \nabla^2 \hat{\phi}_1 = -k^2 \hat{\phi}_1 \) on the top and bottom plates, and \( \nabla^2 \hat{\phi}_1 = (k^2 - m^2 / R^2) \hat{\phi}_1 \) on the radial walls, we then have

\[
\begin{align*}
\hat{u}_{i,r} |_{r=R} &= \frac{1}{T} \sqrt{\nu_i / |\omega|} (k^2 + m^2 / R^2) \hat{\phi}_1 |_{r=R}, \quad \tag{D.12a} \\
\hat{u}_{1,z} |_{z=H_1} &= \frac{1}{T} \sqrt{\nu_i / |\omega|} k^2 \hat{\phi}_1 |_{z=H_1}, \quad \tag{D.12b} \\
\hat{u}_{2,z} |_{z=-H_2} &= \frac{1}{T} \sqrt{\nu_i / |\omega|} k^2 \hat{\phi}_2 |_{z=-H_2}. \quad \tag{D.12c}
\end{align*}
\]

The power dissipated in the boundary layers is then

\[
\begin{aligned}
D &= 2\pi R \left[ \int_0^{H_1} \hat{\rho}_1 \hat{u}_{1,r} |_{r=R} dz + \int_{-H_2}^{0} \hat{\rho}_2 \hat{u}_{2,r} |_{r=R} dz \right] + \\
&\quad + 2\pi \int_0^R \hat{\rho}_1 \hat{u}_{1,z} |_{z=H_1} dr + 2\pi \int_0^R \hat{\rho}_2 (-\hat{u}_{2,z}) |_{z=-H_2} dr.
\end{aligned}
\tag{D.13}
\]

Physically, we expect that \( \text{Re}(D) \geq 0 \), since only then there is damping. Using \( \hat{\rho}_i = \rho_i (i \omega) \hat{\phi}_i \), we can now estimate how each surface contributes to the power dissipation

\[
\begin{align*}
D_{\text{side}} &= -2\pi \Gamma \left( k^2 - m^2 / R^2 \right) \left[ \sqrt{\nu_1 / |\omega|} \rho_1 R \int_0^{H_1} |\hat{\phi}_1 |_{r=R}^2 dz \right. \\
&\quad \left. + \sqrt{\nu_2 / |\omega|} \rho_2 R \int_{-H_2}^0 |\hat{\phi}_2 |_{r=R}^2 dz \right] , \quad \tag{D.14a} \\
D_{\text{top}} &= 2\pi \Gamma k^2 \left[ \sqrt{\nu_1 / |\omega|} \rho_1 R \int_0^R |\hat{\phi}_1 |_{z=H_1}^2 r dr \right] , \quad \tag{D.14b} \\
D_{\text{bot}} &= 2\pi \Gamma k^2 \left[ \sqrt{\nu_2 / |\omega|} \rho_2 R \int_0^R |\hat{\phi}_2 |_{z=-H_2}^2 r dr \right] . \quad \tag{D.14c}
\end{align*}
\]

The contributions from the top and the bottom lids clearly have the right sign, but this is not the case of \( D_{\text{side}} \). With \( k^2 - (m^2 / R^2) > 0 \), this calculation gives \( D_{\text{side}} < 0 \), which is unphysical, but nevertheless a mathematical fact.
The above problem has its origin in the contact line region. Since we do not describe correctly the boundary layers near the contact line, we certainly miss some essential terms that can balance the unphysical term $D_{\text{side}}$. We do not want to calculate these complex layers, but we can isolate the problematic contribution in $D_{\text{side}}$ that arises near the meniscus. Using $\hat{p}_i^* = \rho_i(i\omega)\hat{\phi}_i^*$ and integrating by parts, we transform (D.11) into

$$\mathcal{D} = \Gamma \sum_{i=1,2} \rho_i \sqrt{\nu_i |\omega|} \int_{\Sigma_i} |\hat{u}_{i,\perp}|^2 dS$$

$$+ \Gamma (2\pi R) \left[ \rho_1 \sqrt{\nu_1 |\omega|} \hat{\phi}_1^* \hat{u}_{1,z} - \rho_2 \sqrt{\nu_2 |\omega|} \hat{\phi}_2^* \hat{u}_{2,z} \right] \bigg|_{(r,z) = (R,0)}.$$  \hfill (D.15)

The term $D_{\text{reg}}$ behaves well since $\text{Re}(D_{\text{reg}}) \geq 0$ always. It is some measure of kinetic energy in the boundary layer region over the solid walls. The term $D_{\text{prob}}$ concentrates all the problematic terms and is located in the region of the contact line $(r, z) = (R, 0)$. This contribution should at least be partly canceled by a better modeling of the boundary layer in this region. In the absence of a better contact line model, we brutally discard the problematic terms $D_{\text{prob}}$ and keep only the well-behaving term $D_{\text{reg}}$ in what follows. This yields a complex frequency shift $\alpha = \lambda_{\text{visc}} + i\delta_{\text{visc}}$ with

$$\lambda_{\text{visc}} = -\frac{\text{Re}(D_{\text{reg}})}{2\kappa}, \quad \delta_{\text{visc}} = -\frac{\text{Im}(D_{\text{reg}})}{2\kappa} = \text{Sgn}(\omega) \lambda_{\text{visc}},$$  \hfill (D.16)

and we obtain the expressions (2.79)-(2.80).

D.2. Second approach: viscous damping using the mechanical energy balance

The formula for the viscous damping rate $\lambda_{\text{visc}}$ may alternatively be obtained using the method of Case & Parkinson (1957) and this is briefly demonstrated here. The method of Case & Parkinson (1957) starts from the mechanical energy balance; that is,

$$\frac{d}{dt} \left( \sum_{i=1,2} \frac{1}{2} \int_{\Omega_i} \rho_i \|u_i\|^2 dV \right) + \frac{d}{dt} \left( \frac{1}{2} \int_{\Sigma} \left( (\rho_2 - \rho_1) g \eta^2 + \gamma_1 [\nabla \eta] |\nabla^2 \eta| \right) dS \right)$$

$$= - \sum_{i=1,2} \rho_i \nu_i \int_{\Omega_i} \| \nabla \times u_i \|^2 dV + \sum_{i=1,2} \rho_i \nu_i \int_{\Sigma} \left[ (\nabla \times u_i) \times u_i \right] \cdot n_i dS.$$  \hfill (D.17)

This equation is derived by integrating over the two fluid regions the momentum equations multiplied by $u_i$, $i \in \{1, 2\}$, using all the boundary conditions together with a few integration by parts. On the left-hand side, we recognize the time-derivative of the mechanical energy. On the right-hand side, we find the viscous dissipation in the bulk of the fluid and a surface integral over the free interface. We posit that damping at the interface is weak and we will ignore this term in what follows; that is, we assume that $D'_{\text{visc}} \approx 0$ up to low order terms. By doing this we also ignore potential problems at the contact line.

We inject in (D.17) the flow profiles of a viscously decaying rotating gravity wave
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\[ \mathbf{u}_i \approx \left( \hat{\mathbf{u}}_i + \tilde{\mathbf{u}}_i \right) e^{i(\omega + \delta_{\text{visc}})t} + \text{c.c.} \]  
\[ \eta \approx \tilde{\eta} e^{i(\omega + \delta_{\text{visc}})t} + \text{c.c.} \]

Here c.c. denotes the "complex conjugated part." It is important to use real-valued fields in this energetic approach, otherwise one can easily obtain damping rates that are wrong by a factor of two (a frequent error in the literature). We calculate the kinetic and potential energy approximatively by ignoring the boundary layer corrections and find

\[ E_{c} \approx \left( \sum_{i=1,2} \int_{V_i} \rho \| \hat{\mathbf{u}}_i \|^2 \ dV \right) e^{2\lambda_{\text{visc}}t} = K e^{2\lambda_{\text{visc}}t}, \]
\[ E_{p} \approx \left( \int_{S} \left( (\rho_2 - \rho_1) g |\tilde{\eta}|^2 + \gamma_1 |\nabla^2 \tilde{\eta}| \right) \ dS \right) e^{2\lambda_{\text{visc}}t} = K e^{2\lambda_{\text{visc}}t}. \]

These equations highlight the physical problems that do arise near the contact line.


Zikanov, O. 2018 Shallow water modeling of rolling pad instability in liquid metal batteries. Theoretical and Computational Fluid Dynamics 32 (3), 325–347.